Betti numbers and regularity of projective monomial curves

by

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Abstract

In this thesis we describe how the balancing of the Tor functor can be used to compute the minimal free resolution of a graded module M over the polynomial ring $B = \mathbb{K}[X_0, \ldots, X_m]$ ($\mathbb{K}$ a field X_i's indeterminates). Using a correspondence due to R. Stanley and M. Hochster, we explicitly show how this approach can be used in the case when $M = \mathbb{K}[S]$, the semigroup ring of a subsemigroup $S \subseteq \mathbb{N}^l$ (containing 0) over $\mathbb{K}$ and when M is a monomial ideal of B.

We also study the class of affine semigroup rings for which $\mathbb{K}[S] \cong B/\mathfrak{p}$ is the homogeneous coordinate ring of a monomial curve in $\mathbb{P}^n_{\mathbb{K}}$. We use easily computable combinatorial and arithmetic properties of S to define a notion which we call stabilization. We provide a direct proof showing how stabilization gives a bound on the $\mathbb{N}$-graded degree of minimal generators of $\mathfrak{p}$ and also show that it is related to the regularity of $\mathfrak{p}$. Moreover, we partition the above mentioned class into three cases and show that this partitioning is reflected in how the regularity is attained. An interesting consequence is that the regularity of $\mathfrak{p}$ can be effectively computed by elementary means.

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Chapter 1

Introduction

Let S be a subsemigroup (containing 0) of $\mathbb{Z}^l$, $l \geq 1$ with minimal generating set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$. Let $\mathbb{K}$ denote a field, $R = \mathbb{K}[S]$ the semigroup ring of S over $\mathbb{K}$ and $B = \mathbb{K}[X_0, \ldots, X_n]$ the polynomial ring over $\mathbb{K}$. In what follows we say that S is an affine semigroup and that R is an affine semigroup ring. We endow an S-grading on B by setting $\deg(X_i) = \mathbf{a}_i$. The semigroup ring R can be identified with the subring of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$ generated by the monomials $\mathbf{T}^b = t_1^{b_1} \ldots t_l^{b_l}$, $b = (b_1, \ldots, b_l) \in S$ which is, tautologically, S-graded by setting $\deg(\mathbf{T}^b) = b$. Let $\mathfrak{p}$ be the kernel of the natural surjection $B \to R$ defined by sending $X_i \mapsto \mathbf{T}^{\mathbf{a}_i}$. Then $B/\mathfrak{p} \cong R$ and it is via this representation that we regard R as a graded B-module. Thus, to R we may associate the invariants $\beta_{i,m} = \dim_{\mathbb{K}} \operatorname{Tor}_i^B(\mathbb{K}, R)_m$, $m \in S$, which we call the *multigraded Betti numbers* of R. These numbers count the number of generators of degree m in the ith step of the minimal free resolution of R as a B-module. These numbers thus help to measure the extent to which R fails to be free over B. If the multigrading is positive, we may coarsen the multigrading to an $\mathbb{N}$-grading, setting $\deg(\mathbf{T}^b) = b_1 + \ldots b_l$ for example, and then consider another invariant, that of *Castelnuovo-Mumford regularity* (or simply *regularity*), which is defined to be $\operatorname{reg}(R) = \max\{b_i - i\}$ where $b_i = \max\{\beta_{i,m} \mid m \in \mathbb{N}\}$. This invariant helps to measure how hard it is to compute the minimal free resolution of R.

In this thesis we study affine semigroup rings and are interested in the extent to which these invariants can be computed by combinatorial and arithmetic means. The correspondence we investigate (Theorem 3.1.3) is due to Stanley, Hochster and perhaps others. This correspondence is well known, see [31, Chapter I Theorem 7.9, p. 49], or [26, Theorem 9.2, p. 175] for example. For a slightly more general correspondence see [12]. On page 49 of [31], Stanley writes that, at that time, no applications of this correspondence had yet been found. Since that account, the extent to which the minimal free resolution of R, or its invariants as defined above, can be computed from this correspondence has been extensively studied. Accounts such as [13] and [5] focus on trying to find minimal ideal generators of $\mathfrak{p}$. On the other hand, accounts such as [14] and [8] consider higher syzygies. In the following chapters we describe such applications.

The outline for this thesis is as follows. In Chapter 2 we provide some background and define some notation and conventions which we use for the remainder of the thesis. In Chapter 3 we establish the correspondence of Theorem 3.1.3 and show how the minimal free resolution of R can be computed from this correspondence. We have not found this done explicitly anywhere in the literature. Moreover, in doing so, we relate [7] and [26]. In Chapters 4 and 5 we study the class of affine semigroups associated to projective monomial curves. We denote this class by $\mathscr{C}'$ and describe it shortly. More specifically, in Chapter 4 we use the methods of [28] and [29] to define the notion of *the integer i for which S has stabilized* (Definition 4.2.15). This notion is used to classify the elements of $\mathscr{C}'$ in terms of easily computable combinatorial and arithmetic properties of S (Theorem 4.2.17). In Section 4.3 we also give a direct proof showing how Theorem 4.2.17 and Definition 4.2.15 are related to the maximum $\mathbb{N}$-graded degree of minimal ideal generators of $\mathfrak{p}$. In Chapter 5 we explore the methods of [14] and show how they are related to the theory of Chapter 4 and Theorem 3.1.3. We also obtain a description of regularity in terms of Definition 4.2.15 and, in Theorem 5.9.1, show how regularity is reflected in terms of Theorem 4.2.17. In Chapter 6 we provide a brief summary of the thesis and make some comments about the results presented.

Before describing the class of semigroups $\mathscr{C}'$ which will be considered in Chapters 4 and 5, we would like to make some comments concerning regularity. More specifically, we would like to mention that it is well known that if $S \in \mathscr{C}'$ then the regularity of $\mathfrak{p}$ is attained in at least one of the last two steps in the minimal free resolution. Moreover, if R is Cohen-Macaulay, regularity is always attained in the last step of the resolution. (This follows from Proposition 1.1 and Corollary 1.2 of [2] or from Exercise 20.19 of [17].) In Theorem 5.9.1 we shed more light onto this phenomenon by clarifying which monomial curves attain regularity in the last step of the minimal free resolution.

We now describe the class of semigroups $\mathscr{C}'$ which will be considered in Chapters 4 and 5. An element $S \in \mathscr{C}'$ is a subsemigroup of $\mathbb{N}^2$ whose minimal generating set $\Lambda = \{\mathbf{a}_0 = (d,0), \mathbf{a}_1 = (d-m_1, m_1), \dots, \mathbf{a}_n = (0,d)\}$ is constructed from a set $\mathscr{S} = \{m_1, \dots, m_n = d\}$, $\gcd(\{m_i\}) = 1$, $0 < m_1 < \cdots < m_n$ of integers. The semigroup ring $R = \mathbb{K}[S]$ can be identified with the monomial subring $\mathbb{K}[s^d, s^{d-m_1}t^{m_1}, \dots, s^{d-m_{n-1}}t^{m_{n-1}}, t^d]$ of the polynomial ring $\mathbb{K}[s,t]$ in two variables. The surjection $B \to R$, defined by sending $X_i \mapsto \mathbf{T}^{\mathbf{a}_i}, \mathbf{T}^b = s^{b_1}t^{b_2}, b \in S$, induces an embedding of $\operatorname{Proj} R$ as a monomial curve $C_{\mathscr{S}}$ in $\mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj} B$. Let $\mathfrak{p}$ be the kernel of this surjection. Then $B/\mathfrak{p} \cong R$ is the homogeneous coordinate ring of $C_{\mathscr{S}}$. Throughout this thesis we informally identify $\mathscr{S}$ with this curve.

1.1 Some notation and conventions

We now make some comments about notation and conventions. The natural numbers $\mathbb{N}$ contain zero, all rings are commutative, $|A|$ denotes the cardinality of a set A and $A \backslash B$ denotes the set theoretic complement of two sets A and B. If A and B are sets we let $A + B = \{x + y \mid x \in A, y \in B\}$, $x + A = \{x + y \mid y \in A\}$ and $-A = \{-x \mid x \in A\}$. We have included an index. Many important symbols etc., are included there with page references indicating where they are defined.

Chapter 2

Literature Review

In this chapter we recall some background from several standard references.

2.1 The main objects of study

The goal of this section is to develop some of the theory of semigroups and semigroup rings. We follow treatments given in [18], [23] and [9].

2.1.1 Semigroups and monoids

Definition 2.1.1. A (commutative) *monoid* is a set S with one commutative, associative operation, $+$, and an identity element 0 (i.e., $0 + s = s + 0 = s$ for all $s \in S$). A commutative monoid S is *cancellative* if $s + t = s + u$ implies that $t = u$ for all $s, t, u \in S$. We say that S is *finitely generated* if there exist a finite number of elements $s_1, \ldots, s_m \in S$ such that for any $s \in S$, we can write $s = a_1 s_1 + a_2 s_2 + \cdots + a_m s_m$, $a_i \in \mathbb{N}$. A non-zero element $s \in S$ is *irreducible* if s cannot be written as the sum of two other non-zero elements of S. An element $s \in S$ is a *unit* if $-s \in S$. We say that

S is *pointed* if its only unit is 0, equivalently, $S \cap -S = \{0\}$.

In this thesis we use the term *semigroup* instead of monoid. Thus, we are assuming that *all semigroups contain* 0. This is to be consistent with accounts such as [26].

Given semigroups $(P, *)$ and $(S, +)$ we say that a function $\phi : P \to S$ is a *semigroup homorphism* if $\phi(0_P) = 0_S$ and $\phi(p_1 * p_2) = \phi(p_1) + \phi(p_2)$ for all $p_1, p_2 \in P$. It is clear that the collection of semigroups and semigroup homomorphisms form a category and that any other reasonable restrictions, such as restricting to commutative and pointed semigroups, result in full subcategories.

For every commutative semigroup $(S, +)$ there exists a unique smallest abelian group in which we can map S via a semigroup homomorphism. We denote this group by $G(S)$ and say that $G(S)$ is the *quotient group* of S. We refer to [23, I.7] for the construction.

There is a homomorphism of semigroups $\iota : S \to G(S)$ which has the following universal property. Given a semigroup homomorphism $\phi : S \to H$ into an abelian group H there exists a unique $\psi : G(S) \to H$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\phi"] & G(S) \arrow[dl, "\psi"] \\ H & \end{tikzcd}$$

commutes. In this situation ι is a universal (repelling) object in the category of homomorphisms of S into abelian groups. (A morphism between two semigroup homomorphisms, $f_1 : S \to A_1$, $f_2 : S \to A_2$, in this category is a group homorphism $g : A_1 \to A_2$, such that $gf_1 = f_2$.) Moreover, if S is cancellative then ι is an inclusion.

2.1.2 Semigroup rings

From now on we assume that S is a commutative, cancellative semigroup with operation $+$ and identity 0. We now let B be a commutative ring and to S we associate, in a natural way, a ring which we denote by $B[S]$. The following construction works in more general settings such as if S and B are both not commutative. See [18] for that situation. If the operation of S is multiplicative we refer to [18] or [23] although this is only a notational change.

Let $B[S]$ denote the collection of functions, $f : S \to B$, which are finitely supprted, i.e., $f(x) = 0$ for all but finitely many $x \in S$. Given $f, g \in B[S]$ define addition and multiplication of functions by the formulas $(f + g)(x) = f(x) + g(x)$ and $(f * g)(x) = \sum_{y+z=x} f(y) * g(z)$, where the symbol $\sum_{y+z=x}$ indicates that the sum is taken over all pairs (y, z) of elements of S such that $y + z = x$ and $(f * g)(x) = 0$ if $x \neq y + z$ for any $y, z \in S$.

It is clear that $f + g \in B[S]$. On the other hand, since there are only finitely many $(y, z) \in S \times S$ such that $f(y) * g(z) \neq 0$, the sum $\sum_{y+z=x} f(y) * g(z)$ is finite. Thus, there are only finitely many expressions $y + z = x$ for $(y, z) \in S \times S$ for which $(f * g)(x) \neq 0$. Hence, $f * g$ is a function $S \to B$ which is finitely non-zero so that $f * g \in B[S]$.

We have shown that the above definitions make $B[S]$ closed under addition and multiplication. The function $1_{B[S]}$, defined by $0 \mapsto 1_B$ and $x \mapsto 0$ for $x \neq 0$, is the multiplicative identity of $B[S]$. It is now straightforward to verify that $B[S]$ is a commutative ring.

We now develop a notation which is convenient since the operation of S is $+$. Our goal is to regard $B[S]$ as a free B-module with basis corresponding to elements of S.

This notation will make it clear that $B[S]$ faithfully encompasses the structure of S and B in the sense that an isomorphic copy of each is embedded in $B[S]$.

Let $x \in S$ and let $r_x \in B$. Let $r_x\mathbf{T}^x : S \to B$ denote the function that sends $x \mapsto r_x$ and $y \mapsto 0$ for all $y \neq x$, $x, y \in S$. By construction, $r_x\mathbf{T}^x \in B[S]$. In particular, the function $1_B\mathbf{T}^x$, which we now simply denote as $\mathbf{T}^x$, is the function that sends $x \mapsto 1_B$, and $y \mapsto 0$, $y \neq x$. Thus, $\mathbf{T}^0$ is the identity of $B[S]$.

Let $f \in B[S]$. We claim that f has a unique expression of the form $f = \sum_{x\in S} r_x\mathbf{T}^x$ such that $x \in S$, $r_x \in B$, and $r_x = 0$ for all but a finite number of x. Indeed, by definition, f is finitely non-zero so that f is determined by those elements $x \in S$ such that $f(x) \neq 0$. Set $r_x = f(x)$ for all $x \in S$ and set $h = \sum_{x\in S} r_x\mathbf{T}^x$. Then, by construction, $h(x) = r_x = f(x)$ for all $x \in S$, so $h = f$. Moreover, the set $\{r_x\}_{x\in S}$ consists of finitely many non-zero elements so that $f = \sum_{x\in S} r_x\mathbf{T}^x$ is a finite sum.

Using our formulas for multiplication in $B[S]$, if $f = \sum_{x\in S} r_x\mathbf{T}^x$ and $g = \sum_{y\in S} r_y\mathbf{T}^y$ then, $f * g$ is given by:

$$(\sum_{x\in S} r_x\mathbf{T}^x)(\sum_{y\in S} r_y\mathbf{T}^y) = \sum_{x,y} r_x r_y \mathbf{T}^{x+y}.$$

In particular, $\mathbf{T}^x\mathbf{T}^y = \mathbf{T}^{x+y}$. We can also "add componentwise" i.e., if $f = \sum_{x\in S} r_x\mathbf{T}^x$, and $g = \sum_{x\in S} \tilde{r}_x\mathbf{T}^x$, then $f + g$ is given by:

$$\sum_{x\in S} r_x\mathbf{T}^x + \sum_{x\in S} \tilde{r}_x\mathbf{T}^x = \sum_{x\in S}(r_x + \tilde{r}_x)\mathbf{T}^x.$$

We can now consider $B[S]$ as the free B-module with basis S via the embedding, as a homomorphism of semigroups, $x \mapsto \mathbf{T}^x$. The action of B is defined by

$$r \cdot f = r \cdot \sum_{x\in S} r_x\mathbf{T}^x = r\mathbf{T}^0 \sum r_x\mathbf{T}^x = \sum r r_x\mathbf{T}^x.$$

The map $B \to B[S]$, given by $r \mapsto r \cdot \mathbf{T}^0$, is also an embedding of B as a homomorphism of rings. Thus, $B[S]$ encodes the structure of B and $B[S]$ is an B-algebra.

Given a homomorphism $\phi : S \to P$ of semigroups there exists a unique homomorphisms of semigroup algebras $h : B[S] \to B[P]$ such that $h(\mathbf{T}^x) = \mathbf{T}^{\phi(x)}$ for all $x \in S$ and $h(r\mathbf{T}^{0_S}) = r\mathbf{T}^{0_P}$ for all $r \in B$ [23, Proposition 3.1, p. 106]. Similarly, given a homomorphism $\psi : B \to A$ of rings and a semigroup S, there is a unique homomorphism $\tilde{h} : B[S] \to A[S]$ such that $\sum_{x\in S} r_x\mathbf{T}^x \mapsto \sum_{x\in S} \psi(r_x)\mathbf{T}^x$ [23, Proposition 3.2, p. 107].

In this way we can verify that, for a fixed B and arbitrary semigroups S and P the assignment $S \mapsto B[S]$, $\phi \in \mathrm{Hom}_{\mathrm{Semigroups}}(S, P) \mapsto h \in \mathrm{Hom}_{B\text{-algebras}}(B[S], B[P])$ is functorial. Similarly, for a fixed semigroup S and an arbitrary rings A and B, we can verify that the assignment

$$B \mapsto B[S],\ \phi \in \mathrm{Hom}_{\mathrm{Rings}}(B, A) \mapsto \tilde{h} \in \mathrm{Hom}_{\text{Semigroup rings over } S}(B[S], A[S])$$

is functorial.

2.1.3 Affine semigroups and semigroup rings

For the remainder of this thesis S denotes a semigroup, $\mathbb{K}$ denotes a field and $R = \mathbb{K}[S]$, the semigroup ring of S over $\mathbb{K}$.

Definition 2.1.2. An *affine semigroup* is a finitely generated subsemigroup (containing 0) of $\mathbb{Z}^l$, $l \geq 1$.

Definition 2.1.2 implies that we may specify an affine semigroup, $S \subseteq \mathbb{Z}^l, l \geq 1$, by giving a finite set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$, $\mathbf{a}_i \in \mathbb{Z}^l$ of generators. Thus, the quotient group

$G(S)$ will be a subgroup of $\mathbb{Z}^l$. In particular, $G(S)$ will be a free abelian group (of rank less than or equal to l).

Using the notation of Section 2.1.2, we have that $R = \mathbb{K}[\mathbf{T}^{\mathbf{a}_0}, \dots, \mathbf{T}^{\mathbf{a}_n}]$. Thus, after using the identification $\mathbf{T}^{\mathbf{a}_j} = t_1^{a_1} \dots t_l^{a_l}$, $\mathbf{a}_j = (a_1, \dots, a_l) \in \mathbb{Z}^l$, we may regard R as a monomial subring (i.e., a subring generated by monomials) of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm 1}, \dots, t_l^{\pm 1}]$. Moreover, $\mathbb{K}[G(S)]$ will also be a monomial subring of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm 1}, \dots, t_l^{\pm 1}]$. We also have, [18, Theorem 21.4] for example, that the Krull dimension of R is equal to that of $\mathbb{K}[G(S)]$, which is equal to the rank of the free abelian group $G(S)$.

2.2 Betti numbers

Let $\Lambda = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}$ be a subset of $\mathbb{Z}^l$, let S be the subsemigroup generated by Λ and let $B = \mathbb{K}[X_0, \dots, X_n]$. Define an S-grading on B by setting $\deg(X_i) = \mathbf{a}_i$. Let M be an S-graded B-module. We assume that this grading is positive, i.e., that one of the following equivalent conditions hold.

Theorem 2.2.1. *[26, Theorem 8.6, p. 151] The following are equivalent for a polynomial ring $B = \mathbb{K}[X_0, \dots, X_n]$ graded by an affine semigroup S.*

1. *There exists an $a \in S$ such that B_a is a finite dimensional $\mathbb{K}$-vector space.*

2. *The only polynomials of degree 0 are constants; i.e., $B_0 = \mathbb{K}$.*

3. *For all $a \in G(S)$ the $\mathbb{K}$-vector space B_a is finite dimensional.*

4. *For all finitely generated S-graded modules M and degrees $a \in G(S)$, the $\mathbb{K}$-vector space M_a is finite dimensional.*

5. *The semigroup S has no units (it is pointed) and no variable X_i has degree zero.*

Consider the following definition.

Definition 2.2.2. We define $\beta_{i,m} = \dim_{\mathbb{K}} \operatorname{Tor}_i^B(\mathbb{K}, M)_m$ for some $m \in S$ to be the ith *Betti number* of M. (We will explain why $\operatorname{Tor}_i^B(\mathbb{K}, M)$ is a graded module shortly.)

Although this is the "right" way to define Betti numbers, a certain amount of explanation is in order, especially if one has not heard of Betti numbers or Tor. We sketch some details now.

A reference for the following discussion is [9, Chapter 6.B]. We first recall some terminology. Let G be a finitely generated free abelian group and let B be a G-graded ring, i.e., we can write $B = \oplus_{i\in G} B_i$ as abelian groups, such that $B_i B_j \subseteq B_{i+j}$. Let M be an B-module. We say that M is G-multigraded if we can write $M = \oplus_{i\in G} M_i$ as abelian groups such that $B_i M_j \subseteq M_{i+j}$. If I is an ideal of B we say that I is multigraded if it is graded as a B-module.

When we talk of mulitigraded rings and modules usually the group G is fixed. Thus, unless stated otherwise, we drop the prefixes and refer simply to a ring or module as being graded.

If f is an element of a graded B-module M then we say that f is *homogeneous of multidegree* i if $f \in M_i$ for some $i \in G$. Sometimes we say simply that f is homogeneous, the degree of f is i, the multidegree of f is i, $\deg(f) = i$, or minor variants thereof.

If M is a graded B-module, then M has a set of homogeneous generators and every element $x \in M$ can be written uniquely as a sum of homogeneous elements $x = \sum_i x_i$, $x_i \in M_i$. If M is a finitely generated B-module, we say that a set of generators for M is *minimal* if the omission of any generator implies that we no longer have a generating set. If M is finitely generated and graded then M has a minimal set of homogeneous generators. If M is a graded module and N is a submodule, then we say that N is a graded submodule if the additive subgroups $N_t = M_t \cap N$ for $t \in G$ define a graded module structure on N.

Let M and N be graded B-modules and consider a B-module homomorphism $\phi : M \to N$. We say that ϕ is graded if there exists $j \in G$ such that $\phi(M_i) \subseteq N_{i+j}$ for all $i \in G$. The *degree* of ϕ is j. The collection of G-graded B-modules and graded homomorphisms of degree 0 form a category which we denote by $\mathcal{M}_0(B)$. For each object M and N of $\mathcal{M}_0(B)$, we denote by $\mathrm{Hom}^0_B(M, N)$, the abelian group of degree zero homomorphisms $M \to N$. (Note that $\mathrm{Hom}^0_B(M, N)$ is not a B-module in the usual way in which $\mathrm{Hom}_B(M, N)$ is.) Moreover, the kernel, image and cokernel of graded homomorphisms are graded as is easily checked using a homogeneous set of generators for the source of the map.

In what follows, we denote by $\mathcal{M}(B)$ the category of B-modules and B-module homorphisms, $\mathscr{C}(\mathcal{M}(B))$ the category of chain complexes of objects of $\mathcal{M}(B)$ and $\mathscr{C}(\mathcal{M}_0(B))$ the category of chain complexes of objects of $\mathcal{M}_0(B)$. Let M be an object of $\mathcal{M}_0(B)$. We denote by $M(j)$ the graded B-module with grading shifted by a factor of $j \in G$. This means that $M(j)_d = M_{j+d}$. In particular, if M is generated in degree 0 then since $M(j)_{-j} = M_0$ we have that $M(j)$ is generated in degree $-j$.

The direct sum, $\oplus_i M_i$, of a collection $\{M_i\}_{i \in I}$ of objects in $\mathcal{M}_0(B)$ is graded. More specifically, $(\oplus_i M_i)_j = \oplus_i M_{i,j}$, where $M_{i,j}$ denotes the degree j^{th} component of M_i. This shows that finite products and coproducts exist in $\mathcal{M}_0(B)$.

Using the above discussion, it is now immediate to verify that $\mathcal{M}_0(B)$ is an abelian category. See [23, Chapter III Section 3, p. 133] for the axioms we must check.

We now show that tensor products of graded objects are graded. We include a proof since the only reference we could find was [11, Exercise 1.5.19 d), p. 39].

Proposition 2.2.3 (Exercise 1.5.19 d), p. 39, [11]). *Let M and N be B-graded modules. Then $M \otimes_B N$ is graded. More specifically, $(M \otimes_B N)_i$ is generated as a $\mathbb{Z}$-module by elementary tensors $m \otimes n$ such that $m \in M_j, n \in N_k$ for all $j, k \in G$ and $j + k = i$.*

Proof. Let M be a graded B-module. Then M has a homogeneous set of generators $\{m_i\}$ such that $\deg m_i = -\beta_i \in G$. Let $\oplus_i B(\beta_i)$ be the free B-module such that each summand $B(\beta_i)$ has basis e_{β_i} with $\deg(e_{\beta_i}) = -\beta_i$. Then $\oplus_i(B(\beta_i))$ is graded by the discussion above. We can form the exact sequence

$$\oplus_i B(\beta_i) \xrightarrow{\psi} M \longrightarrow 0$$

where ψ is a degree zero map sending $e_{\beta_i} \mapsto m_i$. Since the kernel of ψ is a graded submodule of $\oplus B(\beta_i)$, we can extend the above exact sequence into another one:

$$\oplus_i B(\alpha_i) \xrightarrow{\phi} \oplus_i B(\beta_i) \xrightarrow{\psi} M \longrightarrow 0$$

where ϕ is a degree zero map, so we have an exact sequence in $\mathcal{M}_0(B)$.

We now make an observation. Since $N(\beta_i)$ is graded, the natural isomorphism $(B(\beta_i) \otimes_B N) \cong N(\beta_i)$ given by $e_{\beta_i} \otimes n \mapsto e_{\beta_i} n$ shows that each summand, $(B(\beta_i) \otimes_B N)$, of $\oplus_i(B(\beta_i) \otimes_B N)$, is graded with $(B(\beta_i) \otimes_B N)_d$ being generated by elementary tensors of the form $r \otimes n$ such that r and n are homogeneous with degree $\deg(r) + \deg(n) = d$.

Since tensor product commutes with direct sums, the natural map $\phi \otimes 1 : \oplus_i(B(\alpha_i) \otimes_B N) \to \oplus_i(B(\beta_i) \otimes_B N)$ is a morphism in $\mathcal{M}_0(B)$. Moreover, applying the functor

$-\otimes_B N$ to the above exact sequence we obtain another exact sequence (in $\mathcal{M}(R)$):

$$\oplus_i(B(\alpha_i)\otimes_B N) \xrightarrow{\phi\otimes 1} \oplus_i(B(\beta_i)\otimes_B N) \xrightarrow{\psi\otimes 1} M\otimes_B N \longrightarrow 0\,.$$

Exactness implies that $M\otimes_B N \cong \oplus_i(B(\beta_i)\otimes_B N)/(\text{image}(\phi\otimes 1))$, which we denote by $(*)$ and claim is graded in the manner desired.

Consider $\text{image}(\phi\otimes 1)$. Fixing a summand, $B(\alpha_i)\otimes_B N$, of $\oplus_i(B(\alpha_i)\otimes_B N)$, we have, by construction that $e_{\alpha_i}\otimes n \mapsto r_{\alpha_i}\otimes n$ where n is a homogenous element of N, and r_{α_i} is a homogeneous generator of $\ker(\psi)$. This implies that $\text{image}(\phi\otimes 1)$ is graded with $(\text{image}(\phi\otimes 1))_d$ being generated by elementary tensors $r\otimes n$ for some homogeneous elements $r\in(\ker(\psi))_l$, and $n\in N_k$ such that $l+k=d$. This shows that $\oplus_i(B(\beta_i)\otimes_B N)/(\text{image}(\phi\otimes 1))$ is graded in the manner desired. The desired grading on $M\otimes_B N$ is now given by the isomorphism $(*)$. $\square$

We have shown that $\mathcal{M}_0(B)$ is closed under tensor products. Let M be a B-module (B and M need not be graded at the moment). Then $-\otimes_B M$ is a right exact functor $\mathcal{M}(B)\to\mathcal{M}(B)$. Moreover, if B and M are graded then $-\otimes_B M$ is a right exact functor $\mathcal{M}_0(B)\to\mathcal{M}_0(B)$. The category $\mathcal{M}_0(B)$ obviously has enough projectives; thus, we may form the left derived functors of $-\otimes_B M$, which we label as $\text{Tor}_i^B(-,M)$ and describe in the following definitions.

Definition 2.2.4. Let M be a B-module. Let

$$\mathcal{F}:\ \cdots \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

be a chain complex of B-modules. Then $\mathcal{F}$ is a *projective resolution* of M if each F_i is a projective module and $\mathcal{F}$ is exact everywhere, i.e., $\ker(\phi_i)=\text{image}(\phi_{i+1})$, except in homological degree 0, where $M\cong F_0/\,\text{image}(\phi_1)$, i.e., $M\cong \text{coker}(\phi_1)$. If each

F_i is a free module, then we say that $\mathcal{F}$ is a *free resolution* of M. If each F_i is a graded projective module, and each ϕ_i is a degree zero map, then we say that $\mathcal{F}$ is a *graded projective resolution*. In a similar manner we define graded free resolutions. Set $M_0 = M$, $M_i = \ker(\phi_{i-1}), i \geq 1$. The modules M_i, $i \geq 0$ are called the *ith syzygy modules* of M.

Remark 2.2.5. We sometimes augment the above chain complex by adding M to the right of F_0 and taking ϕ_0 to be the natural surjection $F_0 \to M$. This makes the above sequence exact everywhere. In this situation we say that M sits in homological degree -1.

Definition 2.2.6. Let M, N be B-modules, and let

$$\mathcal{F} : \cdots \longrightarrow F_n \xrightarrow{\phi_n} \cdots \longrightarrow F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

be a free resolution of a B-module M. (If we are working in $\mathcal{M}_0(B)$, then each ϕ_i is a degree zero map, and each F_i is a graded free B-module.) The left derived functor of $- \otimes_B N$, which we denote by $\operatorname{Tor}_i^B(M, N)$, can be computed by taking the ith homology module of the complex:

$$\mathcal{F} \otimes_B N : \ldots \longrightarrow \quad F_n \otimes_B N \longrightarrow \cdots \longrightarrow F_0 \otimes_B N \longrightarrow 0.$$

i.e., $\operatorname{Tor}_i^B(M, N) = \ker(\phi_i \otimes 1)/\operatorname{image}(\phi_{i+1} \otimes 1)$, and in particular, $\operatorname{Tor}_i^B(M, N)$ is a B-module. (If B, M and N are graded then so is $\operatorname{Tor}_i^B(M, N)$.)

It is natural to ask whether $\operatorname{Tor}_i^B(N, M) \cong \operatorname{Tor}_i^B(M, N)$ i.e., if the result of applying $- \otimes_B M$ to a free resolution of N and then taking homology, is the same as the result by applying $- \otimes_B N$ to a free resolution of M and then taking homology. This turns out to be the case as we will see shortly. Moreover, as with any left derived functor in an abelian category with enough projectives, $\mathrm{Tor}_i^B(M,N)$ does not depend on the chosen projective resolution [33, Lemma 2.4.1, p.44], for example. This implies that if B is graded and if M and N are graded modules then $\mathrm{Tor}_i^B(M,N)$ is graded. Moreover, this grading is independent on the choice of graded free resolution of M or N.

We now have developed enough terminology to understand Definition 2.2.2. If $B = \mathbb{K}[X_0, \ldots, X_n]$ is positively S-graded (in the sense of Theorem 2.2.1) and if M is a graded B-module, then $B_0 = \mathbb{K}$ and so $\beta_{i,m}$ is the dimension of vector space of the mth graded piece of the module $\mathrm{Tor}_i^B(\mathbb{K}, M)$. Moreover, since the grading of $\mathrm{Tor}_i^B(\mathbb{K}, M)$ is independent of the choice of resolution of M, we have that $\beta_{i,m}$ is an invariant of M.

There is another interpretation of $\beta_{i,m}$ which we now describe. One definition of a *minimal free resolution* for M is a resolution $\mathcal{F}$ of M such that applying $- \otimes_B \mathbb{K}$ to $\mathcal{F}$ turns all of the maps into the zero map. See [17, p.476-477] for example. (It also turns out that a minimal free resolution is unique up to isomorphism of chain complexes. See [17, Section 20.1, p. 494].) In a free resolution of M each F_i is of the form $F_i = \oplus_j B(-b_j)$. Thus, in computing $\mathrm{Tor}_i^B(M, \mathbb{K}) \cong \mathrm{Tor}_i^B(\mathbb{K}, M)$ we might as well choose a minimal free resolution for M. Thus,

$$\mathrm{Tor}_i^B(\mathbb{K}, M)_m \cong \mathrm{Tor}_i^B(M, \mathbb{K})_m \cong (\oplus_j \mathbb{K}(-b_j))_m$$

so that $\beta_{i,m}$ counts the number of generators of degree m of F_i. Equivalently, we may say that $\beta_{i,m}$ counts the number of i-syzygies of degree m. If I is a graded ideal of B, we may choose to resolve I as a B-module, or we may resolve the module B/I. In this case $\mathrm{Tor}_{i-1}^B(\mathbb{K}, I)_m \cong \mathrm{Tor}_i^B(\mathbb{K}, B/I)_m$, $i \geq 1$ so that the Betti numbers of I as a B-module appear in homological degree one less than they appear in a resolution of B/I.

We should also recall one last notion. The *projective dimension* of M as a B-module, which we denote by $\operatorname{pd}_B M$, is the smallest $j \geq 0$ (we set $\operatorname{pd}_B M = \infty$ if no such j exists) such that there exists a projective resolution:

$$0 \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of M as a B-module.

With our current assumptions on B, Hilbert's Syzygy Theorem, [33, Corollary 4.3.8, p. 102] shows that every B-module has finite projective dimension. This, combined with the fact that B is Noetherian, implies that every finitely generated graded B-module has only finitely many non-zero Betti numbers.

2.3 Some combinatorial notions

We now review some combinatorial notions. There are several references such as [26, Chapter 1] or [11, Chapter 5].

2.3.1 Simplicial complexes

Definition 2.3.1. A *simplicial complex* Δ on the vertex set $\{1, \ldots, n\}$ is a collection of subsets, called faces, closed under taking subsets. i.e., if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. A face $\sigma \subseteq \Delta$ with cardinality $|\sigma| = i + 1$ has dimension i. A maximal face (under inclusion) of a simplicial complex is called a *facet*.

It is immediate that a simplicial complex is determined by its facets. Two trivial, although important, examples of simplicial complexes are: $\{\}$, the void complex (i.e., the simplicial complex with no faces), and $\{\emptyset\}$, the irrelevant complex (i.e., the simplicial complex with one face in diemension -1).

Let Δ be a simplicial complex on the vertex set $\{1, \ldots, n\}$. For each integer $i \geq -1$ let $\tilde{C}_i(\Delta)$ be the vector space over a field $\mathbb{K}$ whose basis elements e_σ correspond to i-dimensional faces $\sigma \in \Delta$. (i.e., $|\sigma| = i + 1$.) Note that we consider the empty set as a face of dimension -1 so that if $\Delta \neq \{\}$ then $\tilde{C}_{-1}(\Delta)$ will have basis $e_\emptyset$.

Remark 2.3.2. In our definition we do not require that for each $i \in \{1, \ldots, n\}$ we have $\{i\} \in \Delta$. This convention is different than accounts such as [11, Definition 5.1.1, p. 207]. (The reason for our definition should become obvious in subsequent chapters.)

Definition 2.3.3. The complex:

$$\tilde{C}.(\Delta; \mathbb{K}) : 0 \longrightarrow \tilde{C}_{n-1}(\Delta) \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_0} \tilde{C}_{-1}(\Delta) \longrightarrow 0$$

is said to be the reduced chain complex of Δ. The maps are defined by setting $\delta_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j, \sigma) e_{\sigma \setminus \{j\}}$, where $\operatorname{sign}(j, \sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set $\sigma \subseteq \{1, \ldots, n\}$, written according to some fixed total ordering. Unless otherwise stated we use increasing order. If $i < -1$ or $i > n - 1$ then $\tilde{C}_i(\Delta) = 0$ and $\delta_i = 0$ by definition.

Remark 2.3.4. We omit the verification that $\tilde{C}.(\Delta; \mathbb{K})$ is a complex. (It is straightforward and along the same lines as the proof of Lemma 2.4.1.)

Definition 2.3.5. For each integer i, the $\mathbb{K}$-vector space

$$\widetilde{H}_i(\Delta; \mathbb{K}) = \ker(\delta_i) / \operatorname{image}(\delta_{i+1})$$

in homological degree i is the called the *ith reduced homology of* Δ *over* $\mathbb{K}$.

When computing $\tilde{C}.(\Delta; \mathbb{K})$ and $\tilde{H}_i(\Delta; \mathbb{K})$ the field is fixed. Hence, we sometimes denote these simply as $\tilde{C}.(\Delta)$ and $\tilde{H}_i(\Delta)$. Moreover, we often just refer to $\tilde{H}_i(\Delta)$ as the ith homology of Δ or the homology of Δ or minor variants thereof. If for a simplicial complex Δ and all $i \geq -1$ we have $\tilde{H}_i(\Delta) = 0$, we say that Δ is an *acyclic simplicial complex*.

Remark 2.3.6. Since we are using reduced homology $\dim_{\mathbb{K}} \tilde{H}_0(\Delta)$ is equal to the number of connected components minus one. (Recall that for $H_0(\Delta)$, the non-reduced homology of Δ, i.e., we do not include the empty set as a -1-dimensional face, $\dim_{\mathbb{K}} H_0(\Delta)$ is equal to the number of connected components.) We should also remark that if $\Delta = \{\emptyset\}$ then $\tilde{H}_{-1}(\Delta) \cong \mathbb{K}$ and is zero for all other values of i. On the other hand, if $\Delta = \{\}$, then $\tilde{H}_i(\Delta) = 0$ for all i.

Example 2.3.7. If Δ is the simplicial complex defined by facets $\{\{1, 2\}, \{2, 3, 4\}, \{5\}\}$, then Δ has one two dimensional face, four one dimensional faces, five zero dimensional faces and, of course, one face in dimension negative one. We can represent Δ geometrically in Figure 2.1.

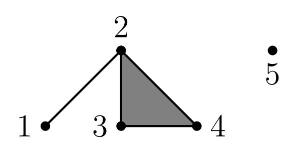


Figure 2.1: The simplicial complex $\Delta = \{\{1, 2\}, \{2, 3, 4\}, \{5\}\}$.

The reduced chain complex of Δ has the form

$$\tilde{C}.(\Delta; \mathbb{K}) : 0 \longrightarrow \tilde{C}_2(\Delta) \xrightarrow{\delta_2} \tilde{C}_1(\Delta) \xrightarrow{\delta_1} \tilde{C}_0(\Delta) \xrightarrow{\delta_0} \tilde{C}_{-1}(\Delta) \longrightarrow 0$$

and $\tilde{C}_2(\Delta)$ has basis $\{e_{\{2,3,4\}}\}$, $\tilde{C}_1(\Delta)$ has basis $\{e_{\{1,2\}}, e_{\{2,3\}}, e_{\{2,4\}}, e_{\{3,4\}}\}$, $\tilde{C}_0(\Delta)$ has basis $\{e_{\{1\}}, \ldots, e_{\{5\}}\}$ and $\tilde{C}_{-1}(\Delta)$ has basis $\{e_\emptyset\}$.

Since δ_2 is an injection, $\tilde{H}_2(\Delta) = 0$. Since $\delta_2(e_{\{2,3,4\}}) = e_{\{3,4\}} - e_{\{2,4\}} + e_{\{2,3\}}$, a basis for image$(\delta_2)$ is given by $\{e_{\{3,4\}} - e_{\{2,4\}} + e_{\{2,3\}}\}$. The map δ_1 can be represented by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the order of the columns correspond to$\{e_{\{1,2\}}, e_{\{2,3\}}, e_{\{2,4\}}, e_{\{3,4\}}\}$, and the order of the rows correspond to $\{e_{\{1\}}, \ldots, e_{\{5\}}\}$. This matrix has rank 3 so the kernel has dimension 1. It is clear that $e_{\{2,3\}} - e_{\{2,4\}} + e_{\{3,4\}}$ maps to zero and so is a basis for $\ker(\delta_1)$. Hence $\tilde{H}_1(\Delta)$ has dimension $1 - 1 = 0$. Similarly, $\tilde{H}_0(\Delta)$ has dimension 1, since $\dim_{\mathbb{K}} \ker(\delta_0) = 4$ and $\dim_{\mathbb{K}} \operatorname{image}(\delta_1) = 3$. It is easy to see that a basis for $\tilde{H}_0(\Delta)$ is given by $\{e_{\{1\}} - e_{\{5\}}\}$. It is in the kernel of δ_0, but is clearly not in the image of δ_1.

As for a combinatorial definition of what it means for Δ to be connected, we have the following, [11, 5.1.26, p. 222]:

Definition 2.3.8. Let Δ be a simplicial complex. We say that Δ is *disconnected* if the vertex set V (i.e., zero dimensional faces) of Δ can be written $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$ and such that no face of Δ has vertices in both V_1 and V_2. If this is not possible, we say that Δ is *connected*.

2.3.2 Some remarks concerning relative homology of simplicial complexes

We now say a few words with respect to the relative homology of simplicial complexes. Let Δ be a simplicial complex on some vertex set V. A *simplicial subcomplex* of Δ is a simplicial complex Γ, such that every face of Γ is a face of Δ. Note that the void complex $\{\}$ is a simplicial subcomplex of every simplicial complex, and that the simplicial complex $\{\emptyset\}$ is a simplicial subcomplex of every simplicial complex except the void complex.

Let $\tilde{C}.(\Delta;\mathbb{K})$ denote the reduced chain complex of Δ and let $\tilde{C}.(\Gamma;\mathbb{K})$ denote the reduced chain complex of Γ. Then, for all $t \geq -1$, we can regard $\tilde{C}_t(\Gamma)$ as a subspace of $\tilde{C}_t(\Delta)$. Moreover, a basis for the quotient $\tilde{C}_t(\Delta,\Gamma) := \tilde{C}_t(\Delta)/\tilde{C}_t(\Gamma)$ can be identified with the symbols e_σ such that σ is a t-dimensional face of Δ and $\sigma \notin \Gamma$. For each $t \geq -1$ let $\delta_t : \tilde{C}_t(\Delta) \to \tilde{C}_{t-1}(\Delta)$ denote the chain complex map of $\tilde{C}.(\Delta;\mathbb{K})$. Define $\bar{\delta}_t : \tilde{C}_t(\Delta,\Gamma) \to \tilde{C}_{t-1}(\Delta,\Gamma)$ to be the image of $\delta_t(e_\sigma)$ in the quotient of $\tilde{C}_t(\Delta)$ by $\tilde{C}_t(\Gamma)$. That $\bar{\delta}_{t-1}\bar{\delta}_t = 0$ follows from the fact that $\delta_{t-1}\delta_t = 0$. Thus, the data $\{\tilde{C}_t(\Delta,\Gamma), \bar{\delta}_t\}_t$ is a chain complex which we denote by $\tilde{C}.(\Delta,\Gamma)$ and call the *relative chain complex* of Δ with respect to Γ. The homology of this complex is called the *relative homology* of Δ by Γ and is denoted by $\tilde{H}_t(\Delta,\Gamma)$.

If $\Gamma = \{\}$ then $\tilde{C}.(\Delta,\Gamma) = \tilde{C}.(\Delta)$. This implies that we will always have $\tilde{C}_{-1}(\Delta,\Gamma) = 0$, unless $\Gamma = \{\}$ and $\Delta = \{\emptyset\}$. Also note that if $\Gamma = \{\emptyset\}$, then $\tilde{H}_0(\Delta,\Gamma) = H_0(\Delta)$.

Given a simplicial subcomplex Γ of Δ, we obtain a short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}.(\Gamma) \longrightarrow \tilde{C}.(\Delta) \longrightarrow \tilde{C}.(\Delta,\Gamma) \longrightarrow 0\ ,$$

and then a long exact sequence

$$\cdots \longrightarrow \tilde{H}_{t+1}(\Delta, \Gamma) \longrightarrow \tilde{H}_t(\Gamma) \longrightarrow \tilde{H}_t(\Delta) \longrightarrow \tilde{H}_t(\Delta, \Gamma) \longrightarrow \cdots$$

of homology. This will become useful Chapter 5.

Let's now consider an example.

Example 2.3.9. Suppose Δ is defined by facets $\{\{1,2\},\{3,4\}\}$ and that Γ is defined by facets $\{\{1,2\},\{4\}\}$. Then, as in the above discussion, $\tilde{C}_t(\Delta, \Gamma) = 0$ for all $t \geq 2$, a basis for $\tilde{C}_1(\Delta, \Gamma)$ is given by $\{e_{\{3,4\}}\}$, a basis for $\tilde{C}_0(\Delta, \Gamma)$ is given by $\{e_{\{3\}}\}$ and $\tilde{C}_{-1}(\Delta, \Gamma) = 0$. Thus $\tilde{C}.(\Delta, \Gamma)$ will take on the form:

$$0 \longrightarrow \tilde{C}_1(\Delta, \Gamma) \xrightarrow{\bar{\delta}_1} \tilde{C}_0(\Delta, \Gamma) \longrightarrow 0\ ,$$

where $\bar{\delta}_1$ is given by sending $e_{\{3,4\}} \mapsto -e_{\{3\}}$. Thus $\tilde{C}.(\Delta, \Gamma)$ will be acyclic, i.e., $\tilde{H}_t(\Delta, \Gamma) = 0$ for all $t \geq -1$.

2.4 The Koszul complex

We now state some facts about the exterior algebra of a free module over a commutative ring and the Koszul complex. These can be found in accounts such as [23].

Let B be a commutative ring and let M be an B-module. We assume the reader is somewhat familiar with the exterior algebra $\bigwedge(M)$. We set $\bigwedge^0(M) = B$. If M is a free B-module with basis $e_1, \ldots, e_n$ then $\bigwedge^p(M) = 0$ for $p > n$. If $1 \leq p \leq n$ then $\bigwedge^p(M)$ is a free B-module, of rank $\binom{n}{p}$, with basis $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ for all p-element subsets $\{i_1, \ldots, i_p\}$ of $\{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_p$.

We can now discuss some facts regarding the Koszul complex. Let $x = x_1, \ldots, x_n$ be a sequence of elements of B. The Koszul complex $\mathcal{K}(x)$ is constructed as follows.

We first set $\mathcal{K}_0 = B$ and define $\mathcal{K}_1$ to be the free B-module F with basis $\{e_1, \ldots, e_n\}$. For $1 < p < n$, we define $\mathcal{K}_p$ to be the free B-module $\bigwedge^p(F)$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$ for all p-element subsets of $\{1, \ldots, n\}$ such that $i_1 < \cdots < i_p$. Finally, we let $\mathcal{K}_n$ to be the free module $\bigwedge^n(F)$ of rank 1 with basis $\{e_1 \wedge \cdots \wedge e_n\}$. For each free module $\mathcal{K}_p$ we define a map δ_p as follows. We set $\delta_1(e_i) = x_i$. For $p > 1$, define $\delta_p : \mathcal{K}_p \to \mathcal{K}_{p-1}$ to be such that $\delta_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}$, where $\widehat{e}_{i_j}$ means that the element e_{i_j} is removed.

We check that $\mathcal{K}(x)$ is a complex.

Lemma 2.4.1. *Let $\mathcal{K}(x)$, $\mathcal{K}_i$, δ_i be defined as above. Then*

$$\mathcal{K}(x):\ 0 \longrightarrow \mathcal{K}_n \xrightarrow{\delta_n} \cdots \longrightarrow \mathcal{K}_1 \xrightarrow{\delta_1} \mathcal{K}_0 \longrightarrow 0$$

is a complex.

Proof. Clearly the composition of two successive maps at the beginning and then end of the sequence is the zero map, so we only need to check that $\delta_p\delta_{p+1}$ is the zero map for $1 \le p \le n-1$. Indeed,

$$\delta_p\delta_{p+1}(e_{i_1} \wedge \cdots \wedge e_{i_{p+1}}) = \delta_p(\sum_{j=1}^{p+1} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_{p+1}})$$

$$= \sum_{j=1}^{p+1} (-1)^{j-1} x_{i_j} \sum_{k=j+1}^{p+1} (-1)^{k-2} x_{i_k} (e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_{p+1}})$$

$$+ \sum_{j=1}^{p+1} (-1)^{j-1} x_{i_j} \sum_{k=1}^{j-1} (-1)^{k-1} x_{i_k} (e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_{p+1}}) = 0,$$

since considering the right hand side, it is clear that all terms cancel in pairs (since $x_{i_j} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_{p+1}}$ appears twice: first with sign $(-1)^{j+k-3}$ and then with opposite sign $(-1)^{j+k-2}$). $\square$

If $B = \mathbb{K}[X_0, \ldots, X_n]$ and $\mathfrak{m} = (X_0, \ldots, X_n)$, we give a combinatorial argument for the following well known statement. We first define some notation. Let $m = (m_0, \ldots, m_n) \in \mathbb{N}^{n+1}$. Define $\operatorname{supp}(m) = \{i \in \{0, \ldots, n\} \mid m_i \neq 0\}$ to be the *support* of m.

Proposition 2.4.2. *Let $B = \mathbb{K}[X_0, \ldots, X_n]$ and let $\mathfrak{m}$ be the maximal ideal $(X_0, \ldots X_n)$. Set $x = X_0, \ldots, X_n$. Then $\mathcal{K}(x)$ is a free resolution of $B/\mathfrak{m}$.*

Proof. Let $\mathbf{e}_i$ denote the standard basis vectors of $\mathbb{N}^{n+1}$ $0 \leq i \leq n$. Give B the standard $\mathbb{N}^{n+1}$-grading by setting $\deg(X_i) = \mathbf{e}_i$. The Koszul complex is also naturally $\mathbb{N}^{n+1}$-graded setting $\deg(e_i) = \mathbf{e}_i$. Since the set $\{\mathbf{e}_0, \ldots, \mathbf{e}_n\}$ is $\mathbb{N}$-linearly independent each $m \in \mathbb{N}^{n+1}$ has a unique expression $m = \sum_{i \in \operatorname{supp}(m)} c_i \mathbf{e}_i$, $c_i > 0$ (if $m = \mathbf{0}$ then $\operatorname{supp}(m) = \emptyset$ and the sum over the empty set is zero). Letting $p = |\operatorname{supp}(m)|$ and restricting $\mathcal{K}(x)$ to degree m, we get a complex of vector spaces:

$$(\mathcal{K})_m : 0 \longrightarrow (\mathcal{K}_p)_m \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_1} (\mathcal{K}_0)_m \longrightarrow 0\,.$$

A $\mathbb{K}$-basis for $(\mathcal{K}_j)_m$, $1 \leq j \leq p$, consists of the $\binom{p}{j}$ elements of the form $\mathbf{X}^{m - \sum_{i \in \sigma} \mathbf{e}_i} e_{k_1} \wedge \cdots \wedge e_{k_j}$, where $\sigma = \{k_1, \ldots, k_j\} \subseteq \operatorname{supp}(m)$ and $k_1 < \cdots < k_j$. Finally, $(\mathcal{K}_0)_m = B_m$ the unique monomial of degree m. (The uniqueness follows from our choice of $\mathbb{N}^{n+1}$-grading.)

Let $\Delta = \{\operatorname{supp}(m)\}$. Then Δ is an aycylic simplicial complex unless $m = \mathbf{0}$, in which case $\Delta = \{\emptyset\}$. Moreover, for all $m \in \mathbb{N}^{n+1}$ there is a canonical isomorphism of complexes:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{K}_p)_m \arrow[r, "\delta_p"] \arrow[d] & \cdots \arrow[r, "\delta_1"] \arrow[d] & (\mathcal{K}_0)_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{C}_{p-1}(\Delta) \arrow[r, "d_{p-1}"] & \cdots \arrow[r, "d_0"] & \tilde{C}_{-1}(\Delta) \arrow[r] & 0.
\end{tikzcd}$$

(This isomorphism is similar to the one given in the proof of Theorem 3.1.3, so we omit some details regarding this isomorphism here.)

Since $(\mathcal{K}_0)_{\mathbf{0}} = 0 \to \mathbb{K} \to 0$, the above isomorphisms implies that $\mathcal{K}(x)$ is acyclic except in homological degree 0. On the other hand, $\operatorname{image} \delta_1 = \mathfrak{m}$ so that $\mathbb{K} = \operatorname{coker}(\delta_1)$ which implies that $\mathcal{K}(x)$ is a free resolution of $\mathbb{K}$. $\square$

Chapter 3

Simplicial complexes and the total tensor resolution

In this chapter we relate several accounts showing how the combinatorial and arithmetic properties of S can be used to explicitly construct the minimal free resolution of $R = \mathbb{K}[S]$.

3.1 A correspondence of Stanley and Hochster

Let S be a subsemigroup of $\mathbb{N}^l, l \geq 1$ and let $\Lambda = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ be a minimal generating set for S. Then $R = \mathbb{K}[S]$ can be identified with the monomial subring $\mathbb{K}[\mathbf{T}^{\mathbf{a}_0}, \ldots, \mathbf{T}^{\mathbf{a}_n}]$ of the polynomial ring $\mathbb{K}[t_1, \ldots, t_l]$, $\mathbf{T}^b = t_1^{b_1} \ldots t_l^{b_l}$, $b = (b_1, \ldots, b_l) \in S$. The ring R is given the tautological S-grading, setting $\deg(\mathbf{T}^b) = b$, for $b \in S$. Let $B = \mathbb{K}[X_0, \ldots X_n]$ and define an S-grading on B by setting $\deg(X_i) = \mathbf{a}_i$. Let $\mathfrak{p}$ be the kernel of the surjective $\mathbb{K}$-algebra homomorphism $B \to R$ sending $X_i \mapsto \mathbf{T}^{\mathbf{a}_i}$. Then $B/\mathfrak{p} \cong R$, an integral domain, so that $\mathfrak{p}$ is a prime ideal.

In Section 2.2 we defined the multigraded Betti numbers of R and $\mathfrak{p}$. We now give a combinatorial interpretation.

Definition 3.1.1 (p. 175, [26])**.** Let S be minimally generated by $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ and let $m \in S$. Define $\Delta_m = \{\sigma \subseteq \{0, \ldots, n\} \mid m - \sum_{i\in\sigma} \mathbf{a}_i \in S\}$.

It is clear from the definition that Δ_m is a simplicial complex, that $\Delta_{\mathbf{0}} = \{\emptyset\}$ and that $\Delta_m \neq \{\}$ for all $m \in S$. In order to compute Δ_m it is necessary to determine semigroup membership. With our assumptions on S we may use the following elementary approach. Let $w := \deg_{\mathbb{N}}(m) = \sum_{i=1}^{l} m_i$, $A := \{\mathbf{0}\}$, $B := \{\}$ and $y := false$. If $w = 0$ then $y := true$. While $\sum_{i=1}^{l} x_i \leq w$ for some $x = (x_1, \ldots, x_l) \in B$ and $y = false$ do: $B := (\Lambda + A)\backslash(A)$; $A := A \cup B$; If $m \in B$ then $y := true$; end do. Return y. Since $S \subseteq \mathbb{N}^l$, adding Λ to a subset $A \subseteq S$ strictly increases the minimum degree, i.e., $\min\{\deg_{\mathbb{N}}(x) \mid x \in \Lambda + A\} > \min\{\deg_{\mathbb{N}}(x) \mid x \in A\}$. This ensures that the process stops. The value of y at the end of the procedure says whether or not $m \in S$. If S is homogeneous (i.e., generated in degree 1) with respect to some $\mathbb{N}$-grading, then the above procedure can be easily modified to be more efficient. The computation of Δ_m is now immediate from the definitions.

Examples 3.1.2. We now illustrate Definition 3.1.1. Let $\Lambda = \{\mathbf{a}_0 = (6,0), \mathbf{a}_1 = (5,1), \mathbf{a}_2 = (1,5), \mathbf{a}_3 = (0,6)\}$ generate S. Then $\Delta_{(6,6)}$ has facets $\{\{0,3\},\{1,2\}\}$, $\Delta_{(5,25)}$ has facets $\{\{2\},\{1,3\}\}$, $\Delta_{(10,20)}$ has facets $\{\{0,2\},\{1,3\}\}$ and $\Delta_{(12,12)}$ is defined by the single facet $\{\{0,1,2,3\}\}$ and is a tetrahedron.

We now make one comment regarding a notational convention. In *Mathematica* the first element of a list is indexed by 1. As such, we have found it convenient, when programing the definition of Δ_m, and all other simplicial complexes which appear in this thesis, to label the vertices from $\{1, \ldots, n+1\}$ as opposed to $\{0, \ldots, n\}$. On the other hand, we use $B = \mathbb{K}[X_0, \ldots, X_n]$, as opposed to $\mathbb{K}[X_1, \ldots, X_{n+1}]$, so that our curves constructed from $\mathscr{S}$, which appear in Chapter 4, are curves in $\mathbb{P}^n$. We use this convention from now on and hope that it does not result in too much confusion.

We now introduce the correspondence of R. Stanley and M. Hochster which we explore in this thesis.

Theorem 3.1.3 (Chapter I Theorem 7.9, p. 49 [31]). *With the notation and assumptions as above, for all i, $0 \leq i \leq n+1$, $\operatorname{Tor}_i^B(\mathbb{K}, R)_m \cong \tilde{H}_{i-1}(\Delta_m)$.*

Proof. Let $x = X_0, \ldots, X_n$. The Koszul complex $\mathcal{K}(x)$ is naturally S-graded, setting $\deg(e_j) = \mathbf{a}_j$. This also implies that the modules $\mathcal{K}_{i+1} \otimes_B R$ are graded. For each i, $-1 \leq i \leq n$, we define $\phi_i : \tilde{C}_i(\Delta_m) \to (\mathcal{K}_{i+1} \otimes_B R)_m$ as follows. For each $\sigma = \{k_1, \ldots, k_{i+1}\} \in \Delta_m$ we send $e_\sigma \mapsto e_{k_1} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{j \in \sigma} \mathbf{a_j}}$, $k_1 < \cdots < k_{i+1}$.

We claim that ϕ_i is an isomorphism of $\mathbb{K}$-vector spaces. For this, we note that a $\mathbb{K}$-basis for $(\mathcal{K}_{i+1} \otimes_B R)_m$ consists of elements of the form $e_{k_1} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{j \in \sigma} \mathbf{a}_j}$, where $\sigma = \{k_1, \ldots, k_{i+1}\} \subseteq \{0, \ldots, n\}$, $k_1 < \cdots < k_{i+1}$ and $m - \sum_{j \in \sigma} \mathbf{a}_j \in S$. On the other hand, $\sigma = \{k_1, \ldots, k_{i+1}\} \in \Delta_m$ if and only if $m - \sum_{j \in \sigma} \mathbf{a}_j \in S$, so that e_σ is a basis vector of $\tilde{C}_i(\Delta_m)$ if and only if $(\mathcal{K}_{i+1} \otimes_B R)_m$ has a basis vector $e_{k_1} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{j \in \sigma} \mathbf{a}_j}$. Thus, ϕ_i is an isomorphism.

To compute $\operatorname{Tor}_i^B(\mathbb{K}, R)$ we apply $- \otimes_B R$ to $\mathcal{K}(x)$ and take homology. Thus, applying $- \otimes_B R$ to $\mathcal{K}(x)$ and restricting to degree m, we claim that the collection $\{\phi_i\}_{i=-1}^n$ yields an isomorphism of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{C}_n(\Delta_m) \arrow[r, "d_n"] \arrow[d, "\phi_n"] & \cdots \arrow[r] \arrow[d] & \tilde{C}_i(\Delta_m) \arrow[r, "d_i"] \arrow[d, "\phi_i"] & \cdots \arrow[r, "d_0"] \arrow[d] & \tilde{C}_{-1}(\Delta_m) \arrow[r] \arrow[d, "\phi_{-1}"] & 0 \; . \\
0 \arrow[r] & (\mathcal{K}_{n+1} \otimes_B R)_m \arrow[r, "\delta_{n+1} \otimes 1"] & \cdots \arrow[r] & (\mathcal{K}_{i+1} \otimes_B R)_m \arrow[r, "\delta_{i+1} \otimes 1"] & \cdots \arrow[r, "\delta_1 \otimes 1"] & (\mathcal{K}_0 \otimes_B R)_m \arrow[r] & 0
\end{tikzcd}$$

Since each ϕ_i is an isomorphism, it remains only to show that the diagram commutes. Let $\sigma = \{k_1, \ldots, k_{i+1}\} \subseteq \{0, \ldots, n\}$ and assume that $k_1 < \cdots < k_{i+1}$. We have,

$$\phi_{i-1}d_i(e_\sigma) = \phi_{i-1}(\sum_{k_j \in \sigma} \operatorname{sign}(k_j, \sigma) e_{\sigma \setminus \{k_j\}})$$

$$= \sum_{k_j \in \sigma} \operatorname{sign}(k_j, \sigma) e_{k_1} \wedge \cdots \wedge \widehat{e}_{k_j} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{l \in \sigma \setminus \{k_j\}} \mathbf{a_l}}.$$

Since k_j is in the jth position of σ (listed in increasing order), $\operatorname{sign}(k_j, \sigma) = (-1)^{j-1}$. On the other hand,

$$(\delta_{i+1} \otimes 1)\phi_i(e_\sigma) = (\delta_{i+1} \otimes 1)(e_{k_1} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{l \in \sigma} \mathbf{a}_l})$$

$$= \sum_{k_j \in \sigma} (-1)^{j-1} X_{k_j} e_{k_1} \wedge \cdots \wedge \widehat{e}_{k_j} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{l \in \sigma} \mathbf{a}_l}$$

$$= \sum_{k_j \in \sigma} (-1)^{j-1} e_{k_1} \wedge \cdots \wedge \widehat{e}_{k_j} \wedge \cdots \wedge e_{k_{i+1}} \otimes \mathbf{T}^{m - \sum_{l \in \sigma \setminus \{k_j\}} \mathbf{a}_l}.$$

Since $\mathcal{K}(x) \otimes_B R$ is a graded complex, $(H_i(\mathcal{K}(x) \otimes_B R))_m = H_i((\mathcal{K}(x) \otimes_B R)_m)$. Moreover, $H_i((\mathcal{K}(x) \otimes_B R)_m)$ is isomorphic to $\tilde{H}_{i-1}(\Delta_m)$ so the result is proved. $\square$

Remark 3.1.4. The isomorphism of complexes proved in the statement is sometimes denoted by $(\mathcal{K}(x) \otimes_B R)_m \cong \tilde{C}.(\Delta_m)[-1]$.

The following corollary is immediate upon noting that $\operatorname{Tor}^B_{i-1}(\mathbb{K}, \mathfrak{p}) \cong \operatorname{Tor}^B_i(\mathbb{K}, R)$, $i \geq 1$.

Corollary 3.1.5 (Theorem 9.2, p. 175, [26])**.** *The Betti number $\beta_{i,m}$ of $\mathfrak{p}$ equals $\dim_{\mathbb{K}} \tilde{H}_i(\Delta_m)$.*

Since for a simplicial complex Δ we have in general $\tilde{H}_0(\Delta) \neq 0$ if and only if Δ is disconnected, the following corollary is immediate.

Corollary 3.1.6. *The ideal $\mathfrak{p}$ has a minimal generator in multidegree m if and only if Δ_m is disconnected.*

Example 3.1.7. The Betti numbers can get quite large even when $S \subseteq \mathbb{N}^2$. Moreover, a single $m \in S$ can contribute to more than one nontrivial Betti number. Let S be generated by

$$\Lambda = \{\{16, 0\}, \{14, 2\}, \{13, 3\}, \{12, 4\}, \{10, 6\}, \{8, 8\}, \{6, 10\}, \{4, 12\}, \{0, 16\}\}$$

and let $m = \{34, 30\}$. Labeling the elements of Λ from 1 through 9, Δ_m is defined by the facets:

$$\{\{3, 8\}, \{1, 5, 8\}, \{2, 4, 8\}, \{2, 5, 9\}, \{2, 6, 7\}, \{2, 6, 8\}, \{2, 7, 9\}, \{3, 6, 9\},$$

$$\{4, 5, 7\}, \{4, 5, 9\}, \{4, 6, 7\}, \{4, 7, 8\}, \{5, 6, 7\}, \{1, 2, 8, 9\}, \{1, 4, 7, 9\},$$

$$\{1, 5, 6, 9\}, \{1, 6, 7, 8\}, \{2, 4, 6, 9\}, \{2, 5, 7, 8\}, \{4, 5, 6, 8\}\}.$$

We then compute that $\dim_{\mathbb{K}} \tilde{H}_1(\Delta_m) = \beta_{1,m} = 1$ and $\dim_{\mathbb{K}} \tilde{H}_2(\Delta_m) = \beta_{2,m} = 11$.

3.2 Some homological notions

We would like to make Theorem 3.1.3 explicit. More specifically, is it possible to compute the minimal free resolution of R as a B-module simply with the knowledge of all Δ_m with $\tilde{H}_i(\Delta_m) \neq 0$, $i \geq -1$? The answer is yes as we will see in Section 3.3. For now, we recall some homological notions which we will need. Let B be a commutative ring. For now we assume we are working in $\mathcal{M}(B)$ and that our complexes are in $\mathscr{C}(\mathcal{M}(B))$. The discussion of Section 2.2 ensures all results will carry over to $\mathcal{M}_0(B)$ and $\mathscr{C}(\mathcal{M}_0(B))$.

For the remainder of this section, we let the complex

$$\mathcal{F} : \cdots \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_1} F_0$$

be a free resolution of a B-module M, and we let the complex

$$\mathcal{G} : \cdots \xrightarrow{\psi_{n+1}} G_n \xrightarrow{\psi_n} \cdots \xrightarrow{\psi_1} G_0$$

be a free resolution of a B-module N. We say that the diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \cdots \\
0 & F_0 \otimes_B G_0 \arrow[l] \arrow[u] & F_0 \otimes_B G_1 \arrow[l] \arrow[u] & F_0 \otimes_B G_2 \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] \\
0 & F_1 \otimes_B G_0 \arrow[l] \arrow[u] & F_1 \otimes_B G_1 \arrow[l] \arrow[u] & F_1 \otimes_B G_2 \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] \\
0 & F_2 \otimes_B G_0 \arrow[l] \arrow[u] & F_2 \otimes_B G_1 \arrow[l] \arrow[u] & F_2 \otimes_B G_2 \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] \\
\vdots & \vdots \arrow[l] \arrow[u] & \vdots \arrow[l] \arrow[u] & \vdots \arrow[l] \arrow[u] & \arrow[l]
\end{tikzcd}$$

is a *double complex* of $\mathcal{F}$ and $\mathcal{G}$, which we denote by $\mathcal{F} \otimes_B \mathcal{G}$. The vertical and horizontal maps are given by $\phi_j \otimes 1 : F_j \otimes_B G_k \to F_{j-1} \otimes_B G_k$ and $(-1)^j \otimes \psi_k : F_j \otimes_B G_k \to F_j \otimes_B G_{k-1}$ respectively.

Since each F_j and G_k is a flat B-module (free modules are flat), the rows and columns of $\mathcal{F} \otimes_B \mathcal{G}$ are all exact except at $F_0 \otimes_B G_k$ vertically and $F_j \otimes_B G_0$ horizontally. It is also evident that each square anti-commutes.

We can make $\mathcal{F} \otimes_B \mathcal{G}$ into a complex, called the *total complex* $\mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})$, as follows. Let $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G}) = \oplus_{j+k=i} F_j \otimes_B G_k$. Then $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$ is a free B-module whose summands correspond to diagonals of the above diagram. For example, $\mathrm{Tot}_1(\mathcal{F} \otimes_B \mathcal{G}) = F_1 \otimes_B G_0 \oplus F_0 \otimes_B G_1$ and in general $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$ has $i+1$ summands. We then define $\mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})$ to be given by the diagram:

$$\cdots \xrightarrow{\delta_{i+1}} \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G}) \xrightarrow{\delta_i} \mathrm{Tot}_{i-1}(\mathcal{F} \otimes_B \mathcal{G}) \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_1} \mathrm{Tot}_0(\mathcal{F} \otimes_B \mathcal{G}) \longrightarrow 0 ,$$

where the differential $\delta_i : \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G}) \mapsto \mathrm{Tot}_{i-1}(\mathcal{F} \otimes_B \mathcal{G})$ is defined in a natural way using the double complex.

We define δ_i by what it does to each $x_{j,k} \in F_j \otimes_B G_k$, such that $j+k=i$. Define $\delta_i(x_{j,k})$ to map into $F_{j-1} \otimes_B G_k$, via the map $x_{j,k} \mapsto (\phi_j \otimes 1)(x_{j,k})$, and also into $F_j \otimes_B G_{k-1}$, via the map $x_{j,k} \mapsto ((-1)^j \otimes_B \psi_k))(x_{j,k})$. Letting $x = x_{i,0} + \cdots + x_{0,i} \in \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$, and letting π_k denote the projection of $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G}) \to F_j \otimes_B G_k$, $0 \leq j,k \leq i$, such that $j+k=i$, we have that $\pi_k\delta_i(x) = (\phi_{j+1} \otimes 1)(x_{j+1,k}) + ((-1)^j \otimes \psi_{k+1})(x_{j,k+1})$. This can be represented by the diagram:

$$\begin{tikzcd}
\pi_k\delta_i(x) = (\phi_{j+1} \otimes 1)(x_{j+1,k}) + ((-1)^j \otimes \psi_{k+1})(x_{j,k+1}) & x_{j,k+1} \in F_j \otimes_B G_{k+1} \arrow[l, "(-1)^j \otimes \psi_{k+1}"'] \\
x_{j+1,k} \in F_{j+1} \otimes_B G_k. \arrow[u, "\phi_{j+1} \otimes 1"]
\end{tikzcd}$$

That $\delta_{i-1}\delta_i(x) = 0$ for all $x = x_{i,0} + \cdots + x_{j,k} + \ldots x_{0,i}, \in \mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})$, $x_{j,k} \in F_j \otimes_B G_k$, $j+k=i$, follows from the fact that $\phi_{j-1}\phi_j = 0$, $\psi_{k-1}\psi_k = 0$, and that the squares anti-commute. Thus, $\mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})$ is a complex.

We have the following statment.

Theorem 3.2.1 (Exercise 1.12, p. 19 [26]). *Let M and N be B-modules and let $\mathcal{F} = \{F_i, \phi_i\}_{i\in\mathbb{N}}$ and $\mathcal{G} = \{G_i, \psi_i\}_{i\in\mathbb{N}}$ be (augmented) free resolutions of M and N respectively. The morphisms $\nu : \mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G}) \to \mathcal{F} \otimes_B N$, where ν_i sends $F_i \otimes_B G_0$ to $F_i \otimes N$ by $1 \otimes \psi_0$ and $F_j \otimes_B G_k \to 0$, $k > 0$, $j+k=i$, and $\eta : \mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G}) \to \mathcal{G} \otimes_B M$ (defined analogously to ν) induce isomorphisms on homology. Moreover, $\eta\nu^{-1}$ induces an explicit isomorphism $\tilde{H}_i(\mathcal{F} \otimes_B N) \cong \tilde{H}_i(M \otimes_B \mathcal{G})$.*

Proof. We prove that ν induces an isomorphism $H_i(\mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})) \cong H_i(\mathcal{F} \otimes_B N)$. Transposing the argument shows that η induces the second isomorphism. That $\eta\nu^{-1}$ induces the final explicit isomorphism is an immediate consequence of the construction of η and ν.

Since the map ψ_0 is the natural surjection $G_0 \to N$, we augment the diagram of $\mathcal{F} \otimes_B \mathcal{G}$ (constructed from the non-augmented resolutions $\mathcal{F}$ and $\mathcal{G}$) using ψ_0. This means that we have the following diagram, where the rows are exact, where the left-most squares commute, and all other squares anti-commute.

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & F_0 \otimes_B N \arrow[l, dashed] \arrow[u, dashed] & F_0 \otimes_B G_0 \arrow[l, dashed, "1\otimes\psi_0"'] \arrow[u] & F_0 \otimes_B G_1 \arrow[l] \arrow[u] & \cdots \arrow[l] \\
0 & F_1 \otimes_B N \arrow[l, dashed] \arrow[u, dashed, "\phi_1\otimes 1"] & F_1 \otimes_B G_0 \arrow[l, dashed, "1\otimes\psi_0"'] \arrow[u] & \cdots \arrow[l] \arrow[u] & \\
0 & F_2 \otimes_B N \arrow[l, dashed] \arrow[u, dashed, "\phi_2\otimes 1"] & \vdots \arrow[l, dashed, "1\otimes\psi_0"'] \arrow[u] & \vdots \arrow[l] \arrow[u] & \\
 & \vdots \arrow[u, dashed] & & &
\end{tikzcd}
$$

We check that ν is a morphism of complexes. Let $x = x_{i,0} + \ldots x_{0,i} \in \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$. Then $(\phi_i \otimes 1) \circ \nu_i(x) = (\phi_i \otimes 1) \circ (1 \otimes \psi_0)(x_{i,0}) = (1 \otimes \psi_0) \circ (\phi_i \otimes 1)(x_{i,0})$ by commutativity of the square. On the other hand, since $\psi_0 \psi_1 = 0$, we have that $\nu_{i-1} \circ \delta_i(x) = (1 \otimes \psi_0)((\phi_i \otimes 1)(x_{i,0}) + ((-1)^{i-1} \otimes \psi_1)(x_{i-1,1}))) = (1 \otimes \psi_0) \circ (\phi_i \otimes 1)(x_{i,0}) = (\phi_i \otimes 1) \circ \nu_i(x)$, so that ν is a morphism of complexes.

We now claim that for each $i \geq 0$, the induced map $\nu_{i*} : H_i(\mathrm{Tot}(\mathcal{F} \otimes \mathcal{G})) \to H_i(\mathcal{F} \otimes_B N)$ is an isomorphism.

We prove that ν_{i*} is surjective, i.e. that every cycle of $F_i \otimes_B N$ is in the image under ν_i of a cycle of $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$, by diagram chasing. Throughout $a \leftarrow b$ means that b maps to a.

Let $x_{i,N} \in F_i \otimes_B N$ be a cycle. Surjectivity of ν_i, commutativity of the left-most squares and exactness of the horizontal differentials implies that we have the following diagram:

$$\begin{tikzcd}
0 & (\phi_i \otimes 1)(b_{i,0}) \arrow[l, "1\otimes\psi_0"'] & \tilde{b}_{i-1,1} \text{ (for some } \tilde{b}_{i-1,1} \in F_{i-1} \otimes_B G_1) \arrow[l, "(-1)^{i-1}\otimes\psi_1"'] \\
x_{i,N} \arrow[u, "\phi_i\otimes 1"] & b_{i,0} \arrow[l, "1\otimes\psi_0"'] \arrow[u, "\phi_i\otimes 1"] &
\end{tikzcd}$$

from which it is clear that taking $b_{i-1,1} = -\tilde{b}_{i-1,1}$ and $b = b_{i,0} + b_{i-1,1}$ implies that $\pi_0\delta_i(b) = 0$, i.e., that $(\phi_i \otimes 1)(b_{i,0}) + ((-1)^{i-1} \otimes \psi_1)(b_{i-1,1}) = 0$, and that $\nu_i(b) = x_{i,N}$, although $\delta_i(b)$ may still be non-zero. This problem can be solved recursively.

Suppose that $b = b_{i,0} + \cdots + b_{j,k} \in \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$, that $\pi_l\delta_i(b) = 0$, $1 \leq l \leq k-1$, and $\nu_i(b) = x_{i,N}$. Then $(\phi_{j+1} \otimes 1)(b_{j+1,k-1}) + ((-1)^j \otimes \psi_k)(b_{j,k}) = 0$, so that $0 = (\phi_j \otimes 1) \circ (\phi_{j+1} \otimes 1)(b_{j+1,k-1}) + (\phi_j \otimes 1) \circ ((-1)^j \otimes \psi_k)(b_{j,k}) = (\phi_j \otimes 1) \circ ((-1)^j \otimes \psi_k)(b_{j,k})$. Thus, we have the diagram:

$$\begin{tikzcd}
0 & \\
(-1)^j \otimes \psi_k(b_{j,k}) \arrow[u, "\phi_j\otimes 1"] & b_{j,k} \arrow[l]
\end{tikzcd}$$

which, combined with anti-commutivity of $\mathrm{Tot}(\mathcal{F} \otimes_B \mathcal{G})$ and exactness of the horizontal differential, implies that we have the following diagram:

$$\begin{tikzcd}
0 & (\phi_j \otimes 1)(-b_{j,k}) \arrow[l] & b_{j-1,k+1} \arrow[l, "(-1)^{j-1}\otimes\psi_{k+1}"'] . \\
& -b_{j,k} \arrow[u, "\phi_j\otimes 1"] &
\end{tikzcd}$$

Considering this diagram, we have that $((-1)^{j-1} \otimes \psi_{k+1})(b_{j-1,k+1}) = (\phi_j \otimes 1)(-b_{j,k})$ so that $(\phi_j \otimes 1)(b_{j,k}) + ((-1)^{j-1} \otimes \psi_{k+1})(b_{j-1,k+1}) = 0$. Hence, taking $b = b_{i,0} + \cdots + b_{j-1,k+1}$, we have that $\pi_k\delta_i(b) = 0$. Thus, continuing this process recursively, we obtain a cycle $b = b_{i,0} + \cdots + b_{0,i}$, mapping to $x_{i,N}$ so that ν_{i*} is surjective.

For injectivity, we show that a cycle of $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$ which maps to a boundary in $F_i \otimes_B N$ is a boundary in $\mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$. For this, let $b = b_{i,0} + \cdots + b_{0,i} \in \mathrm{Tot}_i(\mathcal{F} \otimes_B \mathcal{G})$ be a cycle such that $\nu_i(b) = x_{i,N} \in \mathrm{image}(\phi_{i+1} \otimes 1)$. We claim that $b \in \mathrm{image}(\delta_{i+1})$. Commutativity of the left-most squares and the fact that $x_{i,N} \in \mathrm{image}(\phi_{i+1} \otimes 1)$ yields the commutative diagram:

$$\begin{tikzcd} x_{i,N} & (\phi_{i+1} \otimes 1)(y_{i+1,0}) \arrow[l, "1\otimes\psi_0"'] \\ y_{i+1,N} \arrow[u, "\phi_{i+1}\otimes 1"] & y_{i+1,0} \arrow[l, "1\otimes\psi_0"] \arrow[u, "\phi_{i+1}\otimes 1"'] \end{tikzcd} .$$

On the other hand, the definition of ν_i implies that $x_{i,N} = (1 \otimes \psi_0)(b_{i,0})$, whereby horizontal exactness yields the diagram:

$$\begin{tikzcd} 0 & b_{i,0} - (\phi_{i+1} \otimes 1)(y_{i+1,0}) \arrow[l, "(-1)^i\otimes\psi_0"'] & y_{i,1} \arrow[l, "(-1)^i\otimes\psi_1"'] \end{tikzcd}$$

so that $b_{i,0} = (\phi_i \otimes 1)(y_{i+1,0}) + (-1)^i \otimes \psi_1)(y_{i,1})$. Taking $y = y_{i+1,0} + y_{i,1}$ we have that $b_{i,0} = \pi_0 \delta_{i+1}(y)$.

Suppose now that $y = y_{i+1,0} + \cdots + y_{j+1,k} + y_{j,k+1} \in \mathrm{Tot}_{i+1}(\mathcal{F} \otimes_B \mathcal{G})$ and $b_{j,l} = \pi_l \delta_{i+1}(y)$ for $0 \leq l \leq k$ and $j = i - l$. We construct $y_{j-1,k+2} \in F_{j-1} \otimes_B G_{k+2}$ such that $b_{j-1,k+1} = \pi_{k+1} \delta_{i+1}(y)$ for $y = y_{i+1,0} + \cdots + y_{j,k+1} + y_{j-1,k+2} \in \mathrm{Tot}_{i+1}(\mathcal{F} \otimes_B \mathcal{G})$.

The assumption that b is a cycle implies that $\pi_k \delta_i(b) = 0$. Thus, $(\phi_j \otimes 1)(b_{j,k}) + ((-1)^{j-1}(b_{j-1,k+1}) = 0$. Setting $a = (\phi_j \otimes 1)(b_{j,k}) \in F_{j-1} \otimes_B G_k$ yields the diagram:

$$\begin{tikzcd} a & -b_{j-1,k+1} \arrow[l, "(-1)^{j-1}\otimes\psi_{k+1}"'] \\ b_{j,k} \arrow[u, "\phi_j\otimes 1"] & \end{tikzcd} .$$

The assumption that $b_{j,k} = \pi_k \delta_{i+1}(y)$ yields the anti-commuative diagram:

$$\begin{tikzcd} a & & (\phi_j \otimes 1)(y_{j,k+1}) \arrow[ll, "(-1)^{j-1}\otimes\psi_{k+1}"'] \\ b_{j,k} - (\phi_{j+1}\otimes 1)(y_{j+1,k}) \arrow[u, "\phi_j\otimes 1"] & & y_{j,k+1} \arrow[ll, "(-1)^{j}\otimes\psi_{k+1}"'] \arrow[u, "\phi_j\otimes 1"'] \end{tikzcd}.$$

Combining the previous two diagrams, along with anti-commutativity of the second, we obtain:

$$-((-1)^{j-1} \otimes \psi_{k+1}) \circ (\phi_j \otimes 1)(y_{j,k+1}) = a = ((-1)^{j-1} \otimes \psi_{k+1})(-b_{j-1,k+1}).$$

It follows that $((-1)^{j-1} \otimes \psi_{k+1})(b_{j-1,k+1} - (\phi_j \otimes 1)(y_{j,k+1})) = 0$. Exactness of the horizontal rows, implies that $b_{j-1,k+1} - (\phi_j \otimes 1)(y_{j,k+1}) = ((-1)^{j-1} \otimes \psi_{k+2})(y_{j-1,k+2})$ for some $y_{j-1,k+2} \in F_{j-1} \otimes_B G_{k+2}$. Thus $b_{j-1,k+1} = (\phi_j \otimes 1)(y_{j,k+1}) + ((-1)^{j-1} \otimes \psi_{k+2})(y_{j-1,k+2})$ as desired. Setting $y = y_{i+1,0} + \cdots + y_{j-1,k+2}$ we have that $b_{j-1,k+1} = \pi_k \delta_{i+1}(y)$. Continuing this process recursively, we obtain $y = y_{i+1,0} + \cdots + y_{0,i+1}$ which maps to b so that b is a boundary.

Thus ν_{i*} is injective. Hence, ν induces an isomorphism of homology. $\square$

Since $\mathrm{Tor}_i^B(M,N) \cong H_i(\mathcal{F} \otimes_B N)$ and $\mathrm{Tor}_i^B(N,M) \cong H_i(\mathcal{G} \otimes_B M) \cong H_i(M \otimes_B \mathcal{G})$, Theorem 3.2.1 has an immediate consequence.

Corollary 3.2.2. *Let B be a commutative ring, and let M,N be B-modules. Then* $\mathrm{Tor}_i^B(M,N) \cong \mathrm{Tor}_i^B(N,M)$.

3.3 The total tensor resolution

We now show how Theorem 3.1.3, Theorem 3.2.1 and Corollary 3.2.2 allow for the computation of the minimal free resolution of $R = \mathbb{K}[S]$ from the correspondence of

Theorem 3.1.3. Our account relates at least two accounts in the literature. More specifically, [7] describes one approach to this problem and appears to be using the double complex without exploiting Theorem 3.2.1. On the other hand, [26] does not say how to construct a minimal free resolution of R as a B-module from Theorem 3.1.3. They do, however, present Corollary 3.2.2 as Exercise 1.12, p. 19.

We first describe the general situation and, in fact, we are being more general than [7]. Let S be a pointed affine semigroup and let $B = \mathbb{K}[X_0, \ldots, X_n]$ be positively S-graded in the sense of Theorem 2.2.1. Let $M \cong B/I$ be an S-graded B-module for some ideal I of B and let $\mathcal{F}$ be a graded free resolution of M. As a consequence of the previous section, we have the following description of the Betti numbers of M:

$$\beta_{i,m} = \dim_{\mathbb{K}} \operatorname{Tor}_i^B(\mathbb{K}, M)_m = \dim_{\mathbb{K}} H_i(\operatorname{Tot}(\mathcal{K}(x) \otimes_B \mathcal{F}))_m = \dim_{\mathbb{K}} \operatorname{Tor}_i^B(M, \mathbb{K})_m, m \in S.$$

We have the following statement.

Theorem 3.3.1. *Let $B = \mathbb{K}[X_0, \ldots, X_n]$ be positively S-graded as above. Let M be a finitely generated S-graded B-module. Suppose that for each $0 \leq i \leq \operatorname{pd} M$ and $m \in S$ we know a $\mathbb{K}$-basis for $\operatorname{Tor}_i^B(\mathbb{K}, M)_m$ (defined as $(H_i(\mathcal{K}(x) \otimes_B M))_m$). Then we can construct a minimal free resolution of M.*

Before proceeding, we would like to explain the hypothesis. Although there are many ways to compute $\operatorname{Tor}_i^B(\mathbb{K}, M)$, we are requiring that it be computed by applying $- \otimes_B M$ to $\mathcal{K}(x)$ and then taking homology. We are then requiring the knowledge of a $\mathbb{K}$-basis for $(H_i(\mathcal{K}(x) \otimes_B M))_m$. For example, in the case where $M = R = \mathbb{K}[S]$, by Theorem 3.1.3, this constitutes knowing $\tilde{H}_{i-1}(\Delta_m)$, which, in Chapter 5, we use [14] to show how this can be done for certain S. After giving the proof we do an explicit example to show how this process can be worked out in practice. We also show how Theorem 3.3.1 can be used to compute the minimal free resolutions of monomial ideals of B.

Proof of Theorem 3.3.1. Assume we have a minimal free resolution

$$0 \longleftarrow M \xleftarrow{\phi_0} F_0 \xleftarrow{\phi_1} \cdots \xleftarrow{\phi_{i-1}} F_{i-1} \xleftarrow{\phi_i} F_i \longleftarrow F_{i+1} \longleftarrow \cdots : \mathcal{F}$$

of M and that we explicitly know both a $\mathbb{K}$-basis for $F_0, \ldots F_{i-1}$ and the map ϕ_j, $j < i$. As a consequence of Theorem 3.2.1 and its proof we have an explicit isomorphism between $\mathrm{Tor}_i^B(\mathbb{K}, M)$ (defined as the ith homology of $\mathcal{K}(x) \otimes_B M)$) and $\mathrm{Tor}_i^B(M, \mathbb{K})$ (defined as the ith homology of $\mathbb{K} \otimes_B \mathcal{F}$), which we now exploit.

Since everything is graded, we may restrict to a multidegree $m \in S$. Thus, let $\overline{g}_1, \ldots, \overline{g}_N$ be a $\mathbb{K}$-basis of $\mathrm{Tor}_i^B(\mathbb{K}, M)_m$. As in the proof of Theorem 3.2.1, we may lift $\overline{g}_1, \ldots, \overline{g}_N$ to a set of cycles $g_1, \ldots, g_N$ of $(\mathrm{Tot}_i(\mathcal{K}(x) \otimes_B \mathcal{F}))_m$ which represent a $\mathbb{K}$-basis of $(H_i(\mathrm{Tot}(\mathcal{K}(x) \otimes_B \mathcal{F}))_m$. Then for each h, $1 \leq h \leq N$, we have $g_h = g_{i,0}^h + \cdots + g_{j,k}^h + \cdots + g_{0,i}^h$, for some $g_{j,k}^h \in \mathcal{K}_j \otimes_B F_k$, such that $j + k = i$.

As in the proof of Theorem 3.2.1, each $g_{0,i}^h \in (\mathcal{K}_0 \otimes_B F_i)_m \cong (F_i)_m$ projects to $\overline{g}_{0,i}^h \in \mathrm{Tor}_i^B(M, \mathbb{K})_m$. Thus, the collection $\{g_{0,i}^1, \ldots, g_{0,i}^N\}$ project to a $\mathbb{K}$-basis $\{\overline{g}_{0,i}^1, \ldots, \overline{g}_{0,i}^N\}$ of $\mathrm{Tor}_i^B(M, \mathbb{K})_m$. In $\mathbb{K} \otimes_B \mathcal{F}$ the maps are zero, so the ith homology of $\mathbb{K} \otimes_B \mathcal{F}$ is $F_i/\mathfrak{m}F_i$. Thus, the collection $\{\overline{g}_{0,i}^1, \ldots, \overline{g}_{0,i}^N\}$ form a $\mathbb{K}$-basis for $(F_i/\mathfrak{m}F_i)_m$; by (graded) Nakayama's Lemma, the collection $\{g_{0,i}^1, \ldots, g_{0,i}^N\}$ form the degree m part of a homogeneous B-basis for F_i.

In the construction of each $g_{0,i}^h$, we map $g_{1,i-1}^h$ to $x_{0,i-1}^h \in \mathcal{K}_0 \otimes_B F_{i-1} \cong F_{i-1}$ and then lift to $g_{0,i}^h \in F_i$ (up to sign):

$$\begin{tikzcd} 0 & x_{0,i-1}^h \arrow[l] & g_{0,i}^h \arrow[l] \\ & g_{1,i-1}^h \arrow[u] & \end{tikzcd}$$

By the way a minimal free resolution of M is constructed, the collection $\{x^h_{0,i-1}\}^N_{h=1}$ form the degree m part of a minimal homogeneous B-generating set for $\ker(\phi_{i-1})$.

Thus, assuming we know the collection $\{x^h_{0,i-1}\}^N_{h=1}$ (one for each $\overline{g}^h$) for all $m \in S$ with $\mathrm{Tor}^i_B(\mathbb{K}, M)_m \neq 0$ explicitly, we can redefine F_i by labeling a basis element, e^h of the appropriate degree for each x^h and map the corresponding element of F_i to x^h, $e^h \mapsto x^h$. This makes the map ϕ_i explicit. Continuing this process recursively, we obtain an explicit description of $\mathcal{F}$.

To see that it is possible to know the collection $\{x^h_{0,i-1}\}^N_{h=1}$ explicitly, we first note that the $\mathbb{K}$-vector spaces $(\mathcal{K}_j \otimes_B F_{k-1})_m$ and $(\mathcal{K}_j \otimes_B F_k)_m$ are finite dimensional. We then note that since we are constructing a basis for F_i and defining the map ϕ_i degree by degree, constructing the diagram:

$$\begin{tikzcd} 0 & x^h_{j,k-1} \arrow[l] & g^h_{j,k} \arrow[l] \\ & g^h_{j+1,k-1} \arrow[u] & \end{tikzcd},$$

is a linear algebra problem. Thus, recursively, we can explicitly compute each $x^h_{0,i-1}$ and $g^h_{0,i}$. $\square$

Definition 3.3.2. We call the resolution described in Theorem 3.3.1 as the *total tensor resolution* of M.

Remark 3.3.3. The advantage to the total tensor resolution is that it always produces minimal free resolutions. This is not true for other combinatorial algorithms; see [26], for example, for a survey of several combinatorial methods for producing (generally non-minimal) free resolutions of monomial ideals and affine semigroup rings.

3.4 Minimal free resolutions of affine semigroup rings

Example 3.4.1. Suppose $\Lambda = \{(4,0),(3,1),(1,3),(0,4)\}$ minimally generates S. We now show how to construct the minimal free resolution of $R = \mathbb{K}[S]$.

The diagram that we are dealing with is the following:

$$\begin{tikzcd}
 & & 0 & 0 & 0 & 0 & \\
 & 0 & \mathbb{K}\otimes_B F_0 \arrow[l, dashed] \arrow[u, dashed] & \mathbb{K}\otimes_B F_1 \arrow[l, dashed] \arrow[u, dashed] & \mathbb{K}\otimes_B F_2 \arrow[l, dashed] \arrow[u, dashed] & \mathbb{K}\otimes_B F_3 \arrow[l, dashed] \arrow[u, dashed] & \cdots \arrow[l, dashed] \\
0 & \mathcal{K}_0\otimes_B R \arrow[l, dashed] \arrow[u, dashed] & \mathcal{K}_0\otimes_B F_0 \arrow[l, dashed] \arrow[u, dashed] & \mathcal{K}_0\otimes_B F_1 \arrow[l] \arrow[u, dashed] & \mathcal{K}_0\otimes_B F_2 \arrow[l] \arrow[u, dashed] & \mathcal{K}_0\otimes_B F_3 \arrow[l] \arrow[u, dashed] & \cdots \arrow[l] \\
0 & \mathcal{K}_1\otimes_B R \arrow[l, dashed] \arrow[u, dashed] & \mathcal{K}_1\otimes_B F_0 \arrow[l, dashed] \arrow[u] & \mathcal{K}_1\otimes_B F_1 \arrow[l] \arrow[u] & \mathcal{K}_1\otimes_B F_2 \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & \\
0 & \mathcal{K}_2\otimes_B R \arrow[l, dashed] \arrow[u, dashed] & \mathcal{K}_2\otimes_B F_0 \arrow[l, dashed] \arrow[u] & \mathcal{K}_2\otimes_B F_1 \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & & \\
0 & \mathcal{K}_3\otimes_B R \arrow[l, dashed] \arrow[u, dashed] & \mathcal{K}_3\otimes_B F_0 \arrow[l, dashed] \arrow[u] & \cdots \arrow[l] \arrow[u] & & &
\end{tikzcd}$$

where the dashed arrows denote the augmentation of the double complex $\mathcal{K}(x)\otimes_B \mathcal{F}$. For each $m \in S$ we also have canonical isomorphism $\tilde{C}.(\Delta_m)[-1] \cong (\mathcal{K}(x)\otimes_B R)_m$ (this is proved in the proof of Theorem 3.1.3). We also have $\mathcal{K}_0 \otimes_B F_j \cong F_j$.

In Chapter 5 we give a systematic approach for finding all $m \in S$ with $\tilde{H}_i(\Delta_m) \neq 0$ for some i (with our current assumptions on S). For now, we assume that this has been accomplished. The calculation is summarized in Table 3.1. (In this example we are labeling the elements of Λ from 1 through 4, and $B = \mathbb{K}[X_1,\ldots,X_4]$.)

We already know what the map $\phi_0 : B \to R$ is. Considering $m = (4,4)$, a cycle (of $\tilde{C}_0(\Delta_m)$) which is not a boundary is $e_{\{2\}} - e_{\{1\}}$ and corresponds to $e_2 \otimes \mathbf{T}^{\mathbf{a}_3} - e_1 \otimes \mathbf{T}^{\mathbf{a}_4} \in$

m	Δ_m	i	$\beta_{i,m}$
(3,9)	$\{\{3\},\{2,4\}\}$	1	1
(4,4)	$\{\{1,4\},\{2,3\}\}$	1	1
(6,6)	$\{\{1,3\},\{2,4\}\}$	1	1
(9,3)	$\{\{2\},\{1,3\}\}$	1	1
(6,10)	{{2,3},{2,4},{1,3,4}}	2	1
(7,9)	{{1,3},{1,2,4},{2,3,4}}	2	1
(9,7)	{{2,4},{1,2,3},{1,3,4}}	2	1
(10,6)	{{1,3},{2,3},{1,2,4}}	2	1
(10,10)	{{2, 3, 4}, {1, 2, 4}, {1, 2, 3}, {1, 3, 4}}	3	1

Table 3.1: The multidegrees of syzygies for $\Lambda = \{(4,0),(3,1),(1,3),(0,4)\}$

$$
\begin{tikzcd}
 & & 0 & \bar{1}_{(4,4)} \arrow[l] \\
 & 0 & X_2X_3 - X_1X_4 \arrow[l] \arrow[u] & 1_{(4,4)} \arrow[l] \arrow[u] \\
0 & e_2 \otimes \mathbf{T}^{\mathbf{a}_3} - e_1 \otimes \mathbf{T}^{\mathbf{a}_4} \arrow[l] \arrow[u] & e_2 \otimes X_3 - e_1 \otimes X_4 \arrow[l] \arrow[u] &
\end{tikzcd}
$$

Figure 3.1: Constructing a 1-syzygy

$(\mathcal{K}_1 \otimes_B R)_{(4,4)}$.

Restricting the above diagram to degree $(4,4)$, yields the situation described in Figure 3.1. Thus, we set $g^1_{0,1} = 1_{(4,4)}$. Then $B \cdot 1_{(4,4)}$ is a summand of $\mathcal{K}_0 \otimes_B F_1 \cong F_1$ and we define a map $B \cdot 1_{(4,4)} \to \mathcal{K}_0 \otimes_B F_0 \cong F_0$ by sending $1_{(4,4)} \mapsto X_2X_3 - X_1X_4$ so that $X_2X_3 - X_1X_4$ is a minimal 1-syzygy of R in multidegree $(4,4)$, (or a minimal 0-syzygy of $\mathfrak{p}$).

In a similar manner, lifting the cycles $e_{\{2\}} - e_{\{1\}}$, $e_{\{1\}} - e_{\{2\}}$ and $e_{\{3\}} - e_{\{2\}}$, (of $\tilde{C}_0(\Delta_m)$) corresponding to $m = (9,3),(6,6)$ and $(3,9)$ produces minimal 1-syzygies $X_2^3 - X_1^2X_3$, $X_2X_3 - X_1X_4$ and $X_3^3 - X_2X_4^2$ respectively so that we can respresent ϕ_1

by the matrix

$$\begin{pmatrix} X_2X_3 - X_1X_4 & X_2^3 - X_1^2X_3 & X_1X_3^2 - X_2^2X_4 & X_3^3 - X_2X_4^2 \end{pmatrix}.$$

We will explicitly construct ϕ_3 momentarily. Before doing so, we should note that we can construct ϕ_2 by lifting the cycles $e_{\{1,3\}} - e_{\{2,3\}} - e_{\{1,2\}}$, $-e_{\{2,4\}} + e_{\{2,3\}} + e_{\{3,4\}}$, $e_{\{1,3\}} - e_{\{2,3\}} - e_{\{1,2\}}$, $e_{\{1,3\}} - e_{\{2,3\}} - e_{\{1,2\}}$, $e_{\{2,3\}} + e_{\{3,4\}} - e_{\{2,4\}}$ (of $\tilde{C}_1(\Delta_m)$) corresponding to $m = (10,6), (9,7), (7,9), (6,10)$ respectively, and we represent ϕ_2 by the matrix:

$$\begin{pmatrix} -X_2^2 & -X_1X_3 & -X_2X_4 & -X_3^2 \\ X_3 & X_4 & 0 & 0 \\ X_1 & X_2 & -X_3 & -X_4 \\ 0 & 0 & X_1 & X_2 \end{pmatrix},$$

where the rows correspond to degrees $(4,4), (9,3), (6,6)$ and $(3,9)$ respectively.

To get a sense for how things work for higher syzygies, we show how to construct ϕ_3. For $m = (10,10)$ a cycle (of $\tilde{C}_2(\Delta_m)$) is $e_{\{2,3,4\}} - e_{\{1,3,4\}} + e_{\{1,2,4\}} - e_{\{1,2,3\}}$ which corresponds to $e_2 \wedge e_3 \wedge e_4 \otimes \mathbf{T}^{(6,2)} - e_1 \wedge e_3 \wedge e_4 \otimes \mathbf{T}^{(5,3)} + e_1 \wedge e_2 \wedge e_4 \otimes \mathbf{T}^{(3,5)} - e_1 \wedge e_2 \wedge e_3 \otimes \mathbf{T}^{(2,6)} \in (\mathcal{K}_3 \otimes_B R)_{(10,10)}$. We now have the diagram given in Figure 3.2. Thus, using our previous ordering with respect to the basis, we can represent ϕ_3' by the matrix:

$$\begin{pmatrix} -X_4 \\ X_3 \\ X_2 \\ -X_1 \end{pmatrix}.$$

Using these matrices, we can check, using [20] for example, that we have obtained maps in the minimal free resolution of R.

$$
\begin{tikzcd}
 & & & & 0 & \bar{1}_{(10,10)} \arrow[l] \\
 & & & 0 & \begin{array}{c} X_3 1_{(9,7)} \\ -X_4 1_{(10,6)} \\ +X_2 1_{(7,9)} \\ -X_1 1_{(6,10)} \end{array} \arrow[l] \arrow[u] & 1_{(10,10)} \arrow[l] \arrow[u] \\
 & & 0 & \begin{array}{c} e_1 \otimes (-X_4 1_{(6,6)} - X_3^2 1_{(4,4)} + X_2 1_{(3,9)}) \\ +e_2 \otimes (X_2 X_4 1_{(4,4)} + X_3 1_{(6,6)} - X_1 1_{(3,9)}) \\ +e_3 \otimes (-X_4 1_{(9,3)} - X_2 1_{(6,6)} + X_1 X_3 1_{(4,4)}) \\ +e_4 \otimes (x_3 1_{(9,3)} - X_2^2 1_{(4,4)} + X_1 1_{(6,6)}) \end{array} \arrow[l] & \begin{array}{c} -e_1 \otimes 1_{(6,10)} \\ +e_2 \otimes 1_{(7,9)} \\ +e_3 \otimes 1_{(9,7)} \\ -e_4 \otimes 1_{(10,6)} \end{array} \arrow[l, "\text{(sign}"', "\text{change)}"] \arrow[u] & \\
 & 0 & \begin{array}{c} e_1 \wedge e_2 \otimes (X_2 X_4^2 - X_3^3) \\ e_1 \wedge e_3 \otimes (-X_1 X_3 X_4 + X_2 X_3^2) \\ e_1 \wedge e_4 \otimes (X_1 X_3^2 - X_2^2 X_4) \\ e_2 \wedge e_3 \otimes (X_2^2 X_4 - X_1 X_3^2) \\ e_2 \wedge e_4 \otimes (-X_2^2 X_3 + X_1 X_2 X_4) \\ e_3 \wedge e_4 \otimes (X_2^3 - X_1^2 X_3) \end{array} \arrow[l] \arrow[u] & \begin{array}{c} -e_1 \wedge e_2 \otimes 1_{(3,9)} \\ e_1 \wedge e_3 \otimes X_3 1_{(4,4)} \\ e_1 \wedge e_4 \otimes 1_{(6,6)} \\ -e_2 \wedge e_3 \otimes 1_{(6,6)} \\ -e_2 \wedge e_4 \otimes X_2 1_{(4,4)} \\ e_3 \wedge e_4 \otimes 1_{(9,3)} \end{array} \arrow[l, "\text{(no sign change)}"'] \arrow[u] & & \\
0 & \begin{array}{c} e_2 \wedge e_3 \wedge e_4 \otimes \mathbf{T}^{(6,2)} \\ -e_1 \wedge e_3 \wedge e_4 \otimes \mathbf{T}^{(5,3)} \\ +e_1 \wedge e_2 \wedge e_4 \otimes \mathbf{T}^{(3,5)} \\ -e_1 \wedge e_2 \wedge e_3 \otimes \mathbf{T}^{(2,6)} \end{array} \arrow[l] \arrow[u] & \begin{array}{c} e_2 \wedge e_3 \wedge e_4 \otimes X_2^2 \\ -e_1 \wedge e_3 \wedge e_4 \otimes X_1 X_3 \\ +e_1 \wedge e_2 \wedge e_4 \otimes X_2 X_4 \\ -e_1 \wedge e_2 \wedge e_3 \otimes X_3^2 \end{array} \arrow[l] \arrow[u] & & &
\end{tikzcd}
$$

Figure 3.2: Constructing a 3-syzygy of $R = \mathbb{K}[S]$.

3.5 Minimal free resolutions of monomial ideals

In order to justify the generality in which we stated Theorem 3.3.1, we provide another concrete situation in which the total tensor resolution can be used to compute the minimal free resolution of a graded B-module. For this, we use the total tensor resolution to compute the minimal free resolution of a monomial ideal I of $B = \mathbb{K}[X_0, \ldots, X_n]$ with respect to the standard $\mathbb{N}^{n+1}$-grading. See [26] for several approaches to compute free (generally non-minimal) resolutions of monomial ideals. We first use [12, Proposition 1.1] to derive [26, Theorem 1.34]. Moreover, neither [12] nor [26] say how to use these statements to compute the minimal free resolution of I as a B-module.

We let $S = \mathbb{N}^{n+1}$ and we let $\mathbf{a}_i$ denote the ith standard basis vector of $\mathbb{N}^{n+1}$ so that $\Lambda = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$. We define momentarily the simplicial complex Γ_m of [12, Proposition 1.1]. In fact, it is the same simplicial complex $K^m(I)$, $m \in \mathbb{N}^{n+1}$ of [26, Definition 1.33, p. 16]. The simplicial complex Δ_m is the usual definition that we have been using although, since $S = \mathbb{N}^{n+1}$, Δ_m will be the simplex on the support, $\mathrm{supp}(m) = \{i \in \{0, \ldots, n\} \mid m_i \neq 0\}$, of $m = (m_0, \ldots, m_n) \in S$. More explicitly, the unique monomial $\mathbf{X}^m = X_0^{m_0} \ldots X_n^{m_n}$ of degree $m = (m_0, \ldots, m_n) \in S$ will have Δ_m defined by as the simplex $\{\mathrm{supp}(m)\}$ and will be acyclic.

Let $m \in S$. The simplicial complex Γ_m is defined by:

$$\Gamma_m = \{\sigma \subseteq \{0, \ldots, n\} \mid \mathbf{X}^{m - \sum_{i \in \sigma} \mathbf{a}_i} \in I\}.$$

It is clear that Γ_m is a simplicial subcomplex of Δ_m. Note also that if $\mathbf{X}^m \notin I$, then $\Gamma_m = \{\}$. By [12, Proposition 1.1], $(\mathcal{K}(x) \otimes_B B/I)_m \cong \tilde{C}.(\Delta_m, \Gamma_m)[-1]$ which implies that $\mathrm{Tor}_i^B(\mathbb{K}, B/I)_m \cong \tilde{H}_{i-1}(\Delta_m, \Gamma_m)$, $i \geq 0$. More explicitly, if $\sigma = \{k_1, \ldots, k_i\} \subseteq$

$\{0, \ldots, n\}$, $\sigma \in \Delta_m$, $\sigma \notin \Gamma_m$, then for all $i \geq 0$, the isomorphism $\tilde{C}.(\Delta_m, \Gamma_m)[-1] \cong (\mathcal{K}(x) \otimes_B B/I)_m$ is given by $\tilde{C}_{i-1}(\Delta_m, \Gamma_m) \to (\mathcal{K}_i \otimes_B B/I)_m$, $e_\sigma \mapsto e_{k_1} \wedge \cdots \wedge e_{k_i} \otimes \overline{\mathbf{X}}^{m - \sum_{j \in \sigma} a_j}$, $k_1 < \cdots < k_i$. (The proof of this correspondence is similar to Theorem 3.1.3 and is omitted.)

The short exact sequence

$$0 \longrightarrow \tilde{C}.(\Gamma_m) \longrightarrow \tilde{C}.(\Delta_m) \longrightarrow \tilde{C}.(\Delta_m, \Gamma_m) \longrightarrow 0$$

gives rise to the long exact sequence in homology:

$$\cdots \longrightarrow \tilde{H}_i(\Gamma_m) \longrightarrow \tilde{H}_i(\Delta_m) \longrightarrow \tilde{H}_i(\Delta_m, \Gamma_m) \xrightarrow{\delta_i} \tilde{H}_{i-1}(\Gamma_m) \longrightarrow \cdots.$$

Since Δ_m is acyclic, we have an explicit isomorphism $\delta_i : \tilde{H}_i(\Delta_m, \Gamma_m) \to \tilde{H}_{i-1}(\Gamma_m)$ for all $i \geq 0$, $m \neq \mathbf{0}$. (The map δ_i is the connecting homomorphism.)

Let $\beta_{i,m} = \operatorname{Tor}_i^B(\mathbb{K}, I)_m$. Then the above observations, combined with the relation $\operatorname{Tor}_i^B(\mathbb{K}, B/I)_m \cong \operatorname{Tor}_{i-1}^B(\mathbb{K}, I)_m$, $i \geq 1$, allow us to derive the relation $\beta_{i,m} = \dim_{\mathbb{K}} \tilde{H}_{i-1}(\Gamma_m)$ which is [26, Theorem 1.34, p. 16].

If $\mathbf{X}^m$ is not the least common multiple of some subset of the minimal monomial generators of I then Γ_m is the cone over some subcomplex (and hence aycylic) ([26, Exercise 1.2, p. 18]). Thus, we have a finite set containing all $m \in S$ with $\tilde{H}_i(\Delta_m, \Gamma_m) \neq 0$ and we can compute a $\mathbb{K}$-basis for $\tilde{H}_i(\Delta_m, \Gamma_m)$ by way of the inverse image of δ_i of a $\mathbb{K}$-basis for $\tilde{H}_{i-1}(\Gamma_m)$. Thus, we can use Theorem 3.3.1 to construct the minimal free resolution of B/I.

Example 3.5.1. We now let $B = \mathbb{K}[x, y, z]$ and we let $I = (x^2, xy, xz, y^2, yz, z^2)$. The complexes Γ_m with nontrivial homology are summarized in Table 3.2.

We know the map $\phi_0 : F_0 = B \to B/I$ and, after labeling a basis for F_1 by $e_1^1, \ldots, e_1^6$ and defining ϕ_1 by $e_1^1 \mapsto x^2, \ldots, e_1^6 \mapsto z^2$, we know F_1 and ϕ_1. We consider

m	Γ_m	i	$\beta_{i,m}$
(0, 0, 2)	$\{\emptyset\}$	-1	1
(0, 1, 1)	$\{\emptyset\}$	-1	1
(0, 2, 0)	$\{\emptyset\}$	-1	1
(1, 0, 1)	$\{\emptyset\}$	-1	1
(1, 1, 0)	$\{\emptyset\}$	-1	1
(2, 0, 0)	$\{\emptyset\}$	-1	1
(0,1,2)	$\{\{2\},\{3\}\}$	0	1
(0,2,1)	$\{\{2\},\{3\}\}$	0	1
(1,0,2)	$\{\{1\},\{3\}\}$	0	1
(1,1,1)	$\{\{1\},\{2\},\{3\}\}$	0	2
(1,2,0)	$\{\{1\},\{2\}\}$	0	1
(2,0,1)	$\{\{1\},\{3\}\}$	0	1
(2,1,0)	$\{\{1\},\{2\}\}$	0	1
(1,1,2)	$\{\{1,2\},\{1,3\},\{2,3\}\}$	1	1
(1,2,1)	$\{\{1,2\},\{1,3\},\{2,3\}\}$	1	1
(2,1,1)	$\{\{1,2\},\{1,3\},\{2,3\}\}$	1	1

Table 3.2: The simplicial complex Γ_m for Example 3.5.1

$m = (1, 1, 1)$ and show how to construct the degree m part of a homogeneous basis for F_2 and the corresponding part of ϕ_2.

We first recall how to lift a $\mathbb{K}$-basis of $\tilde{H}_t(\Gamma_m)$ to a $\mathbb{K}$-basis of $\tilde{H}_{t+1}(\Delta_m, \Gamma_m)$ by way of the inverse image of the connecting morphism. For this, we treat a cycle z of $\tilde{C}_t(\Gamma)$ as a cycle in $\tilde{C}_t(\Delta_m)$ by inclusion. We then lift z to an element z' of $\tilde{C}_{t+1}(\Delta_m)$, and then we project z' to obtain the desired cycle, $\overline{z'}$ in $\tilde{C}_{t+1}(\Delta_m, \Gamma_m)$.

A $\mathbb{K}$-basis for $\tilde{H}_0(\Gamma_{(1,1,1)})$ can be identified with the set of cycles $\{-e_{\{1\}}+e_{\{2\}}, -e_{\{1\}}+e_{\{3\}}\} \subset \tilde{C}_0(\Gamma_{(1,1,1)})$. Using the above description, we obtain cycles $\{\overline{e}_{\{1,2\}}, \overline{e}_{\{1,3\}}\}$, of $\tilde{C}_1(\Delta_{(1,1,1)}, \Gamma_{(1,1,1)})$, which correspond to a $\mathbb{K}$-basis of $\tilde{H}_0(\Delta_{(1,1,1)}, \Gamma_{(1,1,1)})$. As before, we then lift to form the degree $(1, 1, 1)$ part of a homogeneous basis for F_2, while in the process defining part of ϕ_2. This procedure is summarized in Figure 3.3, where we produced two basis elements, e_2^1 and e_2^2 of F_2 and they map under ϕ_2 to $ye_1^3 - xe_1^5$

$$\begin{tikzcd}
 & & 0 & \{\overline{e_2^1}, \overline{e_2^2}\} \arrow[l] \arrow[d, leftarrow] \\
 & 0 & \{ye_1^3 - xe_1^5, ze_1^2 - xe_1^5\} \arrow[l] \arrow[u] & \{e_2^1, e_2^2\} \arrow[l] \\
0 & \{xze_2 - zye_1, xye_3 - yze_1\} \arrow[l] \arrow[u] & \{e_1^3 e_2 - e_1^5 e_1, e_1^2 e_3 - e_1^5 e_1\} \arrow[l] \arrow[u] & \\
\{\overline{e}_{\{1,2\}}, \overline{e}_{\{1,3\}}\} \arrow[u] & \{ze_1 \wedge e_2, ye_1 \wedge e_3\} \arrow[l] \arrow[u] & &
\end{tikzcd}$$

Figure 3.3: Forming part of a minimal free resolution for B/I.

and $ze_1^2 - xe_1^5$ respectively.

Chapter 4

Affine semigroups and monomial curves

In this chapter we study the class of affine semigroups for which $R = \mathbb{K}[S] \cong B/\mathfrak{p}$ is the homogeneous coordinate ring of a projective monomial curve. We explore the interplay between properties of S, R and the minimal ideal generators of $\mathfrak{p}$.

4.1 The construction

In what follows we use the methods and terminology of [30], [28] and [29] to study R. We describe these now. Let $\mathscr{S} = \{m_1, \ldots, m_n = d\}$ be a sequence of integers such that $0 < m_1 < \cdots < m_n$, and $\gcd(\{m_i\}) = 1$. To $\mathscr{S}$ we associate a pointed affine semigroup $S \subseteq \mathbb{N}^2$ with minimal generating set $\{(d, 0), (d - m_1, m_1), \ldots, (d - m_{n-1}, m_{n-1}), (0, d)\}$. To S we associate the semigroup ring $R = \mathbb{K}[S]$, for some field $\mathbb{K}$. For ease of notation, we label the $n+1$ generators of S by setting $\mathbf{a}_0 = (d, 0)$, $\mathbf{a}_i = (d - m_i, m_i), 1 \leq i \leq n$, so that $\mathbf{a}_n = (0, d)$. This leads to the natural identification $R \cong \mathbb{K}[\mathbf{T}^{\mathbf{a}_0}, \ldots, \mathbf{T}^{\mathbf{a}_n}]$, where $\mathbf{T}^b = s^{b_1}t^{b_2}$, for some $b = (b_1, b_2) \in S$. As in [29, Definition 2.2, p. 174] we define the dual of $\mathscr{S}$ to be $\hat{\mathscr{S}} = \{d - m_{n-1}, \ldots, d - m_1, d\}$. Let Γ be the numerical semigroup generated by $\mathscr{S}$ and let $\hat{\Gamma}$ be the numerical semigroup generated by $\hat{\mathscr{S}}$. Then Γ is the projection of S onto the second coordinate and $\hat{\Gamma}$ is the projection of S onto the first coordinate. Moreover, there is no loss in generality in assuming that $\gcd(\{m_i\}) = 1$ since otherwise we can divide all elements by their greatest common divisor and obtain an isomorphic semigroup. We denote by $G(S)$ the quotient group of S, which is equal to $\{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv 0 \bmod d\}$ and has rank two. We also have $S_{\mathscr{S}} \cong S_{\hat{\mathscr{S}}}$ (by interchanging coordinates) and thus $\mathbb{K}[S_{\mathscr{S}}] \cong \mathbb{K}[S_{\hat{\mathscr{S}}}]$.

4.1.1 Gradings on the semigroup ring, and Hilbert functions

We consider two gradings on R. The first is the tautological S-grading, setting $\deg(\mathbf{T}^{\mathbf{a}_i}) = \mathbf{a}_i$. The second is a standard $\mathbb{N}$-grading, defined by setting $\deg(s^x t^y) = (x+y)/d$. The $\mathbb{N}$-grading also allows us to view $G(S)$ and S as being $\mathbb{N}$-graded. This is done by setting $\deg((x, y)) = (x+y)/d$ for all $(x, y) \in G(S)$, so that the ith graded piece of S is the set $S_i = \{(x, y) \in S \mid \deg(x, y) = i\}$.

We now briefly recall some facts about Hilbert functions of $\mathbb{N}$-graded $\mathbb{K}$-algebras. Let $A = \oplus_{i \geq 0} A_i$ be an $\mathbb{N}$-graded $\mathbb{K}$-algebra such that A is finitely generated as an $A_0 = \mathbb{K}$-algebra. Define $H_A(i) = \dim_{\mathbb{K}} A_i$ to be the Hilbert function of A. The difference sequence ΔH_A is defined to be $\Delta H_A(i) = H_A(i) - H_A(i-1), i \geq 1, \Delta H_A(0) = 1$. Recall that if $x \in A_1$ is a non-zero divisor of A then $\Delta H_A(i) = H_{A/xA}(i)$.

4.1.2 The $\mathbb{K}$-algebras $\mathrm{gr}(\mathscr{S})$, $\mathrm{gr}(\hat{\mathscr{S}})$

We now describe the $\mathbb{K}$-algebras $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$. The $\mathbb{K}$-algebra $\mathrm{gr}(\mathscr{S})$ appears in [30]; the dual, $\mathrm{gr}(\hat{\mathscr{S}})$, appears in [29]. We do the construction for $\mathrm{gr}(\mathscr{S})$. The $\mathbb{K}$-algebra $\mathrm{gr}(\hat{\mathscr{S}})$ is constructed by transposition.

Let Θ_i be the set of all sums (repetitions allowed) of i elements of $\mathscr{S}$, and let $\mathfrak{M}_i = \Theta_i \setminus \cup_{j<i} \Theta_j$. If $n \in \Gamma$, define $\mathrm{ord}_{\mathscr{S}}(n)$ to be the unique integer i such that $n \in \mathfrak{M}_i$. We define an $\mathscr{S}$-expression of n to be a way of writing n as the sum of elements of $\mathscr{S}$. It follows that $\mathrm{ord}_{\mathscr{S}}(n)$ is the smallest cardinality of an $\mathscr{S}$-expression of n.

Let $\mathrm{gr}(\mathscr{S})_i$ denote the $\mathbb{K}$-vector space with basis $\{t^n \mid n \in \mathfrak{M}_i\}$ and let $\mathrm{gr}(\mathscr{S}) = \oplus_{i\geq 0} \mathrm{gr}(\mathscr{S})_i$. Defining multiplication as:

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } \mathrm{ord}_{\mathscr{S}}(a+b) = \mathrm{ord}_{\mathscr{S}}(a) + \mathrm{ord}_{\mathscr{S}}(b) \\ 0 & \text{otherwise} \end{cases}$$

makes $\mathrm{gr}(\mathscr{S})$ into a ring.

When the set $\mathscr{S}$ is understood, we sometimes drop the subscript and simply write $\mathrm{ord}(n)$. Setting $\deg(t^x) = \mathrm{ord}(x)$ makes $\mathrm{gr}(\mathscr{S})$ standard graded (i.e., $\mathbb{N}$-graded and generated in degree 1) and, by construction, $H_{\mathrm{gr}(\mathscr{S})}(i) = |\mathfrak{M}_i|$. In what follows we let t^d denote the map $\mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ given by $t^x \mapsto t^x t^d$. We also let s^d denote the obvious transposition to $\hat{\mathscr{S}}$. The following was shown in [30].

Proposition 4.1.1 (Theorem 3 a), p. 305, [30]). *If i is sufficiently large then $H_{\mathrm{gr}(\mathscr{S})}(i) = |\mathfrak{M}_i| = d$ and the map $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ is an isomorphism.*

We also have an isomorphism $\mathrm{gr}(\mathscr{S}) \cong R/(s^d)R$, as graded rings. More explicitly, if $\mathrm{ord}_{\mathscr{S}}(x) = n$ then the class of $s^{nd-x}t^x$ in $R/(s^d)R$ corresponds to $t^x \in \mathrm{gr}(\mathscr{S})$.

In fact, for all $j \geq 0$ there exists a natural identification of a $\mathbb{K}$-basis of $\mathrm{gr}(\mathscr{S})_j$ with the elements $A_j = \{x \in S_j \mid x - (d, 0) \notin S\}$, of S_j, corresponding to a $\mathbb{K}$-basis for $(R/(s^d)R)_j$. Explicitly, fix $j \geq 0$ and let $n \in \mathfrak{M}_j$. Then $\mathrm{ord}_{\mathscr{S}}(n) = j$, and n is the sum of the second coordinates of j minimal generators of S. Hence, $x = (jd - n, n) \in S_j$, and we claim that $x \in A_j$. Suppose not. Then $x - (d, 0) \in S$ so that $x - (d, 0) = ((d-1)j - n, n) \in S_{j-1}$, which implies that n is the sum of the second coordinates of $j - 1$ generators of S. In particular $\mathrm{ord}_{\mathscr{S}}(n) \neq j$ so that $n \notin \mathfrak{M}_j$. This a contradiction. Conversely, let $x = (m, n) \in A_j$. Then $(m+n)/d = j$ so that $m = jd - n$. If $\mathrm{ord}(n) < j$ then there exists an $\mathscr{S}$-expression for n containing less than j elements of $\mathscr{S}$ which implies that $(kd - n, n) \in S$ for some $k < j$ so $(kd - n, n) + (j - k)(d, 0) = (m, n) = x$. Thus, if $k < j$ then $x - (d, 0) \in S$, a contradiction. We have shown that the correspondence $\overline{s^{nd-x}t^x} \mapsto t^x$ is bijective. We omit the verification that this is a (degree preserving) ring homorphism. In a similar way $\mathrm{gr}(\hat{\mathscr{S}}) \cong R/(t^d)R$ and we identify a $\mathbb{K}$-basis of $\mathrm{gr}(\hat{\mathscr{S}})_j$ with the elements $C_j = \{x \in S_j \mid x - (0, d) \notin S\}$, for all $j \geq 0$, of S_j. Note that $A_j \cap C_j \neq \emptyset$, in general.

Since R is an integral domain, t^d is a non-zero divisor in R. Moreover, $t^d \in R_1$ so that $H_{\mathrm{gr}(\mathscr{S})}(i)$ is the (first) difference sequence of $H_R(i)$. In particular, $H_{\mathrm{gr}(\mathscr{S})}(i) = \Delta H_R(i) = H_R(i) - H_R(i-1)$. A simple argument by induction shows that $H_R(i) = \sum_{j=0}^{i} H_{\mathrm{gr}(\mathscr{S})}(j) = \sum_{j=0}^{i} |\mathfrak{M}_j|$.

The following was shown in [28].

Lemma 4.1.2 (Lemma 1.4. p. 277,[28])**.** *If $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ is onto (equivalently every element of $\mathfrak{M}_{i+1}$ is the sum of d and an element of $\mathfrak{M}_i$) and $|\mathfrak{M}_{i+1}| = d$ then $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ is an isomorphism for all $j \geq i+1$. Equivalently adding d gives a bijection from $\mathfrak{M}_j$ to $\mathfrak{M}_{j+1}$ for all $j \geq i+1$.*

Proposition 4.1.1 shows that the hypothesis of Lemma 4.1.2 are satisfied for some $i \gg 0$. It is also easy to compute the smallest i satisfying the hypothesis of Lemma 4.1.2. Once i has been determined it is easy to compute H_R. Indeed, we have that $H_{\mathrm{gr}(\mathscr{S})}(j+1) = H_{\mathrm{gr}(\mathscr{S})}(j) = d$ for all $j \geq i+1$. In terms of R, there exists an i such that $H_R(j+1) = H_R(j) + d$ for all $j \geq i$. Thus, computing $|\mathfrak{M}_j|$, $0 \leq j \leq i$, produces the desired result.

The above discussion also carries over for the dual $\mathrm{gr}(\hat{\mathscr{S}})$ although it is not clear, a priori, that the map $s^d : \mathrm{gr}(\hat{\mathscr{S}})_i \to \mathrm{gr}(\hat{\mathscr{S}})_{i+1}$ becomes onto at the same time as $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$. It is not true, in general, that if $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ is onto then $\mathfrak{M}_{i+1} = d$. This will become apparent in examples.

We now illustrate the above discussion with some examples.

Example 4.1.3. Let $\mathscr{S} = \{1, 3, 4\}$. Then S is minimally generated by

$$\Lambda = \{\{4, 0\}, \{3, 1\}, \{1, 3\}, \{0, 4\}\}.$$

We summarize the sets $\mathfrak{M}_i$ and the Hilbert function $H_{\mathrm{gr}(\mathscr{S})}(i)$, $0 \leq i \leq 3$ in Table 4.1. To illustrate multiplication in $\mathrm{gr}(\mathscr{S})$ we have that $t^1 t^3 = 0$ since $\mathrm{ord}_{\mathscr{S}}(1+3) = \mathrm{ord}_{\mathscr{S}}(4) = 1$ whereas $\mathrm{ord}_{\mathscr{S}}(1) + \mathrm{ord}_{\mathscr{S}}(3) = 2$. On the other hand, $t^1 t^4 = t^5$ since $\mathrm{ord}_{\mathscr{S}}(4+1) = \mathrm{ord}_{\mathscr{S}}(5) = 2$, which equals $\mathrm{ord}_{\mathscr{S}}(1) + \mathrm{ord}_{\mathscr{S}}(4)$.

It is clear that adding $\mathfrak{M}_3 \xrightarrow{+4} \mathfrak{M}_4$ is a bijection. Hence, we also have that the Hilbert function of $\mathrm{gr}(\mathscr{S})$ is given by $\{1, 3, 5, 4 \to\}$. This implies that the Hilbert function of $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$ is given by $\{1, 4, 9, 13, 17, 21, \ldots\}$.

Example 4.1.4. It is possible for the Hilbert function of $\mathrm{gr}(\mathscr{S})$ to become constant, i.e., $H_{\mathrm{gr}(\mathscr{S})}(j) = H_{\mathrm{gr}(\mathscr{S})}(k)$ for all $k \geq j$, before the map becomes onto. For example, let $\mathscr{S} = \{5, 9, 11, 20\}$. The sets $\mathfrak{M}_i$ and the Hilbert function $H_{\mathrm{gr}(\mathscr{S})}(i)$, for $0 \leq i \leq 6$, are summarized in Table 4.2.

i	$\mathfrak{M}_i$	$H_{\mathrm{gr}(\mathscr{S})}(i)$
0	$\{0\}$	1
1	$\{1,3,4\}$	3
2	$\{2,5,6,7,8\}$	5
3	$\{9,10,11,12\}$	4
4	$\{13,14,15,16\}$	4

Table 4.1: The data associated to Example 4.1.3

i	$\mathfrak{M}_i$	$H_{\mathrm{gr}(\mathscr{S})}(i)$
0	$\{0\}$	1
1	$\{5,9,11,20\}$	4
2	$\{10,14,16,18,22,25,29,31,40\}$	9
3	$\{15,19,21,23,27,30,33,34,36,38,42,45,49,51,60\}$	15
4	$\{24,26,28,32,35,39,41,43,44,47,50,53,54,56,58,62,65,69,71,80\}$	20
5	$\{37,46,48,52,55,59,61,63,64,67,70,73,74,76,78,82,85,89,91,100\}$	20
6	$\{57,66,68,72,75,79,81,83,84,87,90,93,94,96,98,102,105,109,111,120\}$	20

Table 4.2: The data associated to Example 4.1.4

Hence, the Hilbert function of $\mathrm{gr}(\mathscr{S})$ is $\{1,4,9,15,20,20,20 \rightarrow\}$. We have that $37-20 = 17 \notin \mathfrak{M}_4$, so adding 20 does not give a surjection $\mathfrak{M}_4 \rightarrow \mathfrak{M}_5$. We do however obtain the desired bijection $\mathrm{gr}(\mathscr{S})_i \rightarrow \mathrm{gr}(\mathscr{S})_{i+1}$ when $i = 5$.

4.1.3 Unstable elements and the basis

In [28] elements $x \in \Gamma$ were defined to be *unstable* if there exists $a \in \mathbb{N}$ such that $\mathrm{ord}_{\mathscr{S}}(ad+x) < a + \mathrm{ord}_{\mathscr{S}}(x)$. If no such a exists then x is said to be stable. This definition is equivalent to saying that x is unstable if and only if t^x is killed in $\mathrm{gr}(\mathscr{S})$ by some power of t^d. In the same way, we refer to elements of $\hat{\Gamma}$ as being stable or unstable. Proposition 4.1.1 and Lemma 4.1.2 imply that there exists an $i \gg 0$ such that $t^d : \mathrm{gr}(\mathscr{S})_j \rightarrow \mathrm{gr}(\mathscr{S})_{j+1}$ is injective for all $j \geq i$. This implies that there are only finitely many unstable elements. Since the property of being unstable depends on $\mathscr{S}$

and $\hat{\mathscr{S}}$ and not Γ nor $\hat{\Gamma}$, we sometimes say x is an unstable element of $\mathscr{S}$, $\hat{\mathscr{S}}$ or even $\mathrm{gr}(\mathscr{S})$ or $\mathrm{gr}(\hat{\mathscr{S}})$. The existence of unstable elements is equivalent to R not being Cohen-Macaulay. Indeed, we will show momentarily that s^d, t^d is a homogeneous system of parameters for R. Moreover, R is Cohen-Macaulay if and only if every (and equivalently one) system of parameters is regular (see [31, Theorem 5.9, p. 41] for example) and there exists unstable elements of Γ if and only if t^d is a zero divisor in $\mathrm{gr}(\mathscr{S})$. Also, $\mathscr{S}$ and $\hat{\mathscr{S}}$ need not have the same number of unstable elements as we will see in Example 4.1.6.

Example 4.1.5. Let $x, y \in \Gamma$. Suppose that $y \equiv x \bmod d$ with $y > x$. Then $y = x + qd$ for some $q \geq 1$. Suppose $\mathrm{ord}(y) \leq \mathrm{ord}(x) + q - 1$. Then x is unstable. Indeed, by assumption, $\mathrm{ord}(y) = \mathrm{ord}(x + qd) < \mathrm{ord}(x) + \mathrm{ord}(qd) = \mathrm{ord}(x) + q$ so that $t^x t^{qd} = 0$.

In [19] the semigroup S' was defined to be $\{x \in G(S) \mid x + a(d,0) \in S$ and $x + b(0,d) \in S$ for some $a, b \geq 0\}$. In [29] it was shown that the second coordinates of elements of $S' \backslash S$ are the unstable elements of Γ and the first coordinates are the unstable elements of $\hat{\Gamma}$. It follows, from previous discussion, that R is Cohen-Macaulay if and only if $S = S'$.

In [29], the set $\tilde{S} = \{x \in G(S) \mid x + (d, 0) \in S$ and $x + (0, d) \in S\}$ was defined and it was shown that $\tilde{S}$ is not a semigroup, in general. We do however have the relations $S \subseteq \tilde{S} \subseteq S' \subseteq G(S) \cap \mathbb{N}^2$, as sets, and that the maximum $\mathbb{N}$-degree of an element of $S' \backslash S$ is equal to the maximum degree of an element of $\tilde{S} \backslash S$.

The elements of A_j with second coordinate an unstable element of $\mathscr{S}$ are of the form $x = (d, 0) + s$ for some $s \in S' \backslash S$ such that $\deg(s) + 1 = j$. Similarly, the elements of C_j with second coordinate an unstable element of $\hat{\mathscr{S}}$ are of the form $x = (0, d) + s$, $s \in S' \backslash S$ such that $\deg(s) + 1 = j$.

Consider the set $\mathcal{B} = \{x \in S \mid x - (d,0) \notin S \text{ and } x - (0,d) \notin S\} \subseteq S$. The set $\mathcal{B}$ was used in accounts such as [29] and [24] to study S and R. At the $\mathbb{K}$-algebra level, $\mathcal{B}$ can be identified with the exponent vectors of the monomials of $R/(t^d, s^d)R$. We call $\mathcal{B}$ the *basis* of S over the set $\{(d,0),(0,d)\}$, suggestive by the fact that the obvious lifting of the monomials in $R/(t^d, s^d)R$ is a finite minimal generating set of R as a module over $\mathbb{K}[s^d, t^d]$. This is shown in [24, Lemma 3.1.1, p. 30] and an alternative argument follows from Corollary 4.2.2. It follows that the set $\mathcal{B}$ is finite. In the literature, see [14] for example, $\mathcal{B}$ is sometimes called the Apery set relative to $\{(d,0),(0,d)\}$. In [24] an algorithm to compute $\mathcal{B}$ is given and we present an alternative method in Proposition 4.2.4. In what follows we let $\mathcal{B}_i$ be the set of elements of $\mathcal{B}$ which have degree i. It follows that $\mathcal{B}_i = A_i \cap C_i \subseteq \mathcal{B}$.

Example 4.1.6. We now do a detailed example intended to illustrated the notions introduced thus far. Let $\mathscr{S} = \{1,7,9\}$, so that $\hat{\mathscr{S}} = \{2,8,9\}$. We summarize the sets $\mathfrak{M}_i$ and $\hat{\mathfrak{M}}_i$ in Table 4.3. The underlined numbers represent the unstable elements of $\mathscr{S}$ and $\hat{\mathscr{S}}$.

i	$\mathfrak{M}_i$	$\hat{\mathfrak{M}}_i$
0	0	0
1	{1,7,9}	{2, 8, 9}
2	{2, 8, 10, 14, 16, 18}	{$\underline{4}$, 10, 11, 16, 17, 18}
3	{$\underline{3}$, 11, 15, 17, 19, 21, 23, 25, 27}	{$\underline{6}$, $\underline{12}$, $\underline{13}$, 19, 20, 24, 25, 26, 27}
4	{$\underline{4}$, $\underline{12}$, 20, 22, 24, 26, 28, 30, 32, 34, 36}	{$\underline{14}$, $\underline{15}$, $\underline{21}$, $\underline{22}$, 28, 29, 32, 33, 34, 35, 36}
5	{$\underline{5}$, $\underline{13}$, 29, 31, 33, 35, 37, 39, 41, 43, 45}	{$\underline{23}$, $\underline{30}$, $\underline{31}$, 37, 38, 40, 41, 42, 43, 44, 45}
6	{$\underline{6}$, 38, 40, 42, 44, 46, 48, 50, 52, 54}	{$\underline{39}$, 46, 47, 48, 49, 50, 51, 52, 53, 54}
7	{47, 49, 51, 53, 55, 57, 59, 61, 63}	{55, 56, 57, 58, 59, 60, 61, 62, 63}

Table 4.3: The data associated to Example 4.1.6.

We have that S is minimally generated by $\Lambda = \{\{9,0\},\{8,1\},\{2,7\},\{0,9\}\}$. The basis $\mathcal{B}$ is given by:

$$\mathcal{B} = \{\{0,0\},\{2,7\},\{4,14\},\{6,21\},\{8,1\},\{10,8\},\{12,15\},$$

$$\{14,22\},\{16,2\},\{24,3\},\{32,4\},\{40,5\},\{48,6\}\}.$$

The elements of $\tilde{S}\backslash S$ are:

$$\{\{15,12\},\{23,13\},\{31,5\},\{39,6\}\}.$$

Figure 4.1 shows how everything corresponds to S graphically in the plane. We have scaled everything so that the vertical and horizontal axes indicate the $\mathbb{N}$-graded degree of S. The large solid dots correspond to the elements of the basis. The open circles are the elements of A_j which are not elements of C_j. The open circles with $\times$ in the middle are the elements of C_j which are not elements of A_j. The small black dots are the remaining elements of S. The elements of $\tilde{S}\backslash S$ are represented by solid black squares.

4.2 Stabilization of projective monomial curves

We would like a systematic way to organize the class of affine semigroups constructed in Section 4.1. For this, let $\mathscr{C}_d$ denote the set of affine semigroups contained in $\mathbb{N}^2_d := \{(x,y) \in \mathbb{N}^2 \mid x+y \equiv 0 \bmod d\}$ with minimal generating set $\Lambda_{\mathscr{S}} = \{(d,0),(d-m_1,m_1),\ldots,(d-m_2,m_2),(0,d)\}$ constructed from a set $\mathscr{S} = \{m_1,\ldots,m_n = d\}$ of nonnegative integers such that $0 < m_1 < \cdots < m_n$ and $\gcd(\{m_i\}) = 1$. Let $\mathscr{C}' = \cup_{d\in\mathbb{N}_{>0}}\mathscr{C}_d$.

We now make some observations regarding $\mathcal{B}$, $\mathrm{gr}(\mathscr{S})$, $\mathrm{gr}(\hat{\mathscr{S}})$, $\tilde{S}\backslash S$ and the unstable elements of $\mathscr{S}$ and $\hat{\mathscr{S}}$. We will use these observations to partition the elements of $\mathscr{C}'$

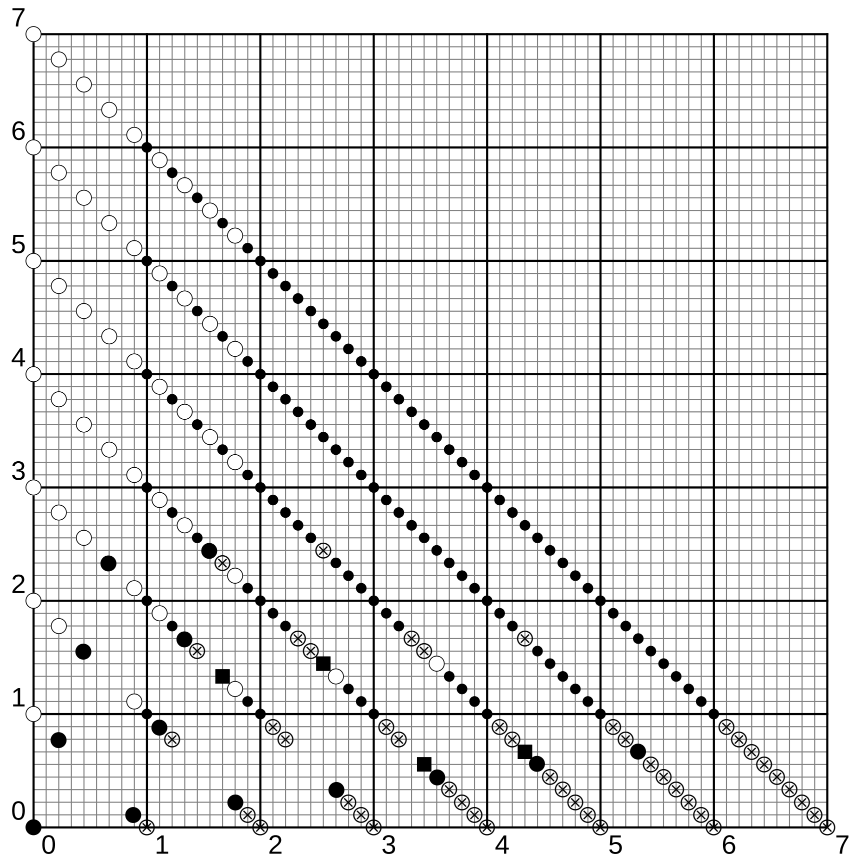


Figure 4.1: $\mathscr{S} = \{1, 7, 9\}$ in the plane

into three classes and relate the theory of this chapter with that of Chapter 3. We first relate the basis $\mathcal{B}$ to $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$.

Theorem 4.2.1. *Let $\mathscr{S}, \hat{\mathscr{S}}, S$ and other notation be as above. The following are equivalent:*

1. *The map $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ is onto.*

2. *The map $s^d : \mathrm{gr}(\hat{\mathscr{S}})_i \to \mathrm{gr}(\hat{\mathscr{S}})_{i+1}$ is onto.*

3. *The basis, $\mathcal{B}$, contains no elements in degree $i+1$.*

4. $S_{i+1} = (S_i + (d, 0)) \cup (S_i + (0, d))$.

Proof. The equivalence of statements 3 and 4 is immediate from the definitions. To see the equivalence of statements 1 and 3, recall the isomorphism $\operatorname{gr}(\mathscr{S}) \cong R/(s^d)R$ so that $\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}) \cong R/(s^d, t^d)R$ and $(\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_{i+1} \cong (R/(s^d, t^d)R)_{i+1}$. The exponent vectors of a $\mathbb{K}$-basis of the right hand side correspond to the elements of $\mathcal{B}$ in degree $i+1$. Since $\deg(t^d) = 1$, the left hand side is equal to $\operatorname{gr}(\mathscr{S})_{i+1}/(t^d)\operatorname{gr}(\mathscr{S})_i$. Having $t^d : \operatorname{gr}(\mathscr{S})_i \to \operatorname{gr}(\mathscr{S})_{i+1}$ onto is equivalent to $\operatorname{gr}(\mathscr{S})_{i+1}/(t^d)\operatorname{gr}(\mathscr{S})_i = 0$, whence statement 1 is equivalent to statement 3. The equivalence of statements 2 and 3 follows similarly. $\square$

Theorem 4.2.1 yields the following corollary.

Corollary 4.2.2. *Suppose one of the equivalent conditions of Theorem 4.2.1 is satisfied in degree i. Then the following equivalent conditions hold.*

1. *The map $t^d : \operatorname{gr}(\mathscr{S})_j \to \operatorname{gr}(\mathscr{S})_{j+1}$ is onto for all $j \geq i$.*
2. *The map $s^d : \operatorname{gr}(\hat{\mathscr{S}})_j \to \operatorname{gr}(\hat{\mathscr{S}})_{j+1}$ is onto for all $j \geq i$.*
3. *The basis $\mathcal{B}$ contains no elements in degree greater than or equal to $i+1$.*
4. *$S_j = (S_{j-1} + (d, 0)) \cup (S_{j-1} + (0, d))$ for all $j \geq i+1$.*
5. *The sets A_j, and C_j, which are the elements of S which correspond to $\mathbb{K}$-bases of $\operatorname{gr}(\mathscr{S})_j$ and $\operatorname{gr}(\hat{\mathscr{S}})_j$, respectively, are disjoint for all $j \geq i+1$.*

Proof. The equivalence of the last three statements are immediate from the definitions and recalling that $\mathcal{B}_i = A_i \cap C_i$. The equivalence of statements 1 and 3 and statements 2 and 3 follow in the same manner as in Theorem 4.2.1. To have no basis elements in degree $i+1$ implies that $R_{i+1}/(s^d, t^d)R_i = 0$. Since $\operatorname{gr}(\mathscr{S})$ is standard graded (i.e.,

$\mathbb{N}$-graded and generated in degree 1), $\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}) \cong R/(s^d, t^d)R$ is standard graded. Thus, $(\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_{i+1} = (\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_1(\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_i$ so that if $(\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_i = 0$, i.e., $t^d : \operatorname{gr}(\mathscr{S})_i \to \operatorname{gr}(\mathscr{S})_{i+1}$ is onto, then we also have $(\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}))_j = 0$ for all $j \geq i$. Thus, statement 3 holds. □

Example 4.2.3. It is possible for the maps $t^d : \operatorname{gr}(\mathscr{S})_j \to \operatorname{gr}(\mathscr{S})_{j+1}$, $s^d : \operatorname{gr}(\hat{\mathscr{S}})_j \to \operatorname{gr}(\hat{\mathscr{S}})_{j+1}$ to become onto in degree $i+1$, without $|\mathfrak{M}_{i+1}| = |\hat{\mathfrak{M}}_{i+1}| = d$ so that $t^d : \operatorname{gr}(\mathscr{S})_j \to \operatorname{gr}(\mathscr{S})_{j+1}$ and $s^d : \operatorname{gr}(\hat{\mathscr{S}})_j \to \operatorname{gr}(\hat{\mathscr{S}})_{j+1}$, are not isomorphisms for all $j \geq i+1$.

All curves up to degree 16 were searched. The first example found was $\mathscr{S} = \{1, 3, 11, 13\}$. The sets $\mathfrak{M}_i$ are summarized in Table 4.4. Considering this example, we see that the map $t^{13} : \operatorname{gr}(\mathscr{S})_4 \to \operatorname{gr}(\mathscr{S})_5$ is onto. On the other hand, the map $t^{13} : \operatorname{gr}(\mathscr{S})_5 \to \operatorname{gr}(\mathscr{S})_6$ is not an isomorphism because $t^{21} \in \ker(t^{13})$.

i	$\mathfrak{M}_i$	$\lvert\mathfrak{M}_i\rvert$
1	{1,3,11,13}	4
2	{2,4,6,12,14,16,22,24,26}	9
3	{5,7,9,15,17,19,23,25,27,29, 33,35,37,39}	14
4	{8,10,18,20,28,30,32,34,36,38,40,42,44,46,48,50,52}	17
5	{21,31,41,43,45,47,49,51,53,55,57,59,61,63,65}	15
6	{54,56,58,60,62,64,66,68,70,72,74,76,78}	13

Table 4.4: The data associated to Example 4.2.3

Below are all such curves of degree 13 that have similar phenomenon.

$$\{\{1, 3, 11, 13\}, \{2, 10, 12, 13\}, \{1, 3, 5, 11, 13\}, \{2, 8, 10, 12, 13\}\}$$

Similarly, below are all such curves of degree 15.

$$\{\{2, 12, 15\}, \{3, 13, 15\}, \{1, 3, 13, 15\}, \{2, 12, 14, 15\}, \{1, 3, 5, 13, 15\},$$
$$\{2, 8, 10, 12, 15\}, \{2, 10, 12, 14, 15\}, \{3, 5, 7, 13, 15\}, \{1, 3, 5, 7, 13, 15\}, \{2, 8, 10, 12, 14, 15\}\}$$

We will momentarily call curves with this phenomenon case 3 curves.

We now show that the basis can be computed from $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$.

Proposition 4.2.4. *Let $x \in \Gamma$ and suppose $\operatorname{ord}(x) = i$. Then $(id - x, x) \in \mathcal{B}_i$ if and only if the canonical image of t^x in $\operatorname{gr}(\mathscr{S})_i/(t^d)\operatorname{gr}(\mathscr{S})_{i-1}$ is non-zero. (The obvious transposition of the statement for $\hat{\mathscr{S}}$ also holds.)*

Proof. This is immediate from the isomorphisms $\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}) \cong R/(t^d, s^d)R \cong \operatorname{gr}(\hat{\mathscr{S}})/(s^d)\operatorname{gr}(\hat{\mathscr{S}})$ of graded rings. □

Remark 4.2.5. Proposition 4.2.4 shows that the basis $\mathcal{B}$ can be computed from the sets $\mathfrak{M}_i$. (The fact that $R/(s^s, t^d)R$ is finite dimensional ensures this is a finite computation.) The advantage is that this algorithm should be faster than computing the basis in S. We also have that the number of basis elements of degree i is equal to $H_{\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S})}(i)$. (This also follows from the isomorphism $\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S}) \cong R/(t^d, s^d)R$.)

A consequence of Corollary 4.2.2 is that the canonical lift of a monomial $\mathbb{K}$-basis of $\operatorname{gr}(\mathscr{S})/(t^d)\operatorname{gr}(\mathscr{S})$ is a minimal finite generating set for $\operatorname{gr}(\mathscr{S})$ as a $\mathbb{K}[t^d]$-module. Similarly, $\operatorname{gr}(\hat{\mathscr{S}})$ is also finitely (and minimally) generated as a $\mathbb{K}[s^d]$-module by the lift of a monomial $\mathbb{K}$-basis of $\operatorname{gr}(\hat{\mathscr{S}})/(s^d)\operatorname{gr}(\hat{\mathscr{S}})$. Motivated by Proposition 4.2.4, we refer to the second coordinates of elements of $\mathcal{B}$ as the "basis" of $\operatorname{gr}(\mathscr{S})$ and the first coordinate of elements of $\mathcal{B}$ as the "basis" $\operatorname{gr}(\hat{\mathscr{S}})$. We make this terminology precise in the following series of definitions. All of these definitions seem to be useful in describing R in what follows.

Definition 4.2.6. Let $x = (x_1, x_2) \in \mathcal{B}$. We say that t^{x_2} is a *basis element* of $\mathrm{gr}(\mathscr{S})$, and that s^{x_1} is a basis element of $\mathrm{gr}(\hat{\mathscr{S}})$. In what follows, we sometimes just refer to x_2 or x_1 as being the basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively.

Definition 4.2.7. If $x = (x_1, x_2) \in \mathcal{B}$ and x_2 is a stable element of $\mathscr{S}$ then we say that t^{x_2} (or sometimes just x_2) is a *stable basis element* of $\mathrm{gr}(\mathscr{S})$. The analogous terminology is used for $\mathrm{gr}(\hat{\mathscr{S}})$.

An equivalent definition for a stable basis element is saying that t^x is a stable basis element of $\mathrm{gr}(\mathscr{S})$ if x is the smallest stable element in its congruence class $\bmod d$. This definition makes it clear that there are exactly d stable basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$, i.e., one per congruence class. Also note that, in general, we need not have $x = (x_1, x_2) \in \mathcal{B}$ with both x_1 and x_2 stable basis elements. One explanation for this is given in Theorem 4.4.4.

Definition 4.2.8. If $x = (x_1, x_2) \in \mathcal{B}$ and x_2 is an unstable element of $\mathscr{S}$ then we say that t^{x_2} is an *unstable basis element* of $\mathrm{gr}(\mathscr{S})$. The analogous terminology is used for $\mathrm{gr}(\hat{\mathscr{S}})$.

We refer to the collection of basis elements of $\mathrm{gr}(\mathscr{S})$ as its basis and the collection of stable basis elements of $\mathrm{gr}(\mathscr{S})$ as its stable basis. Moreover, the stable basis of $\mathrm{gr}(\mathscr{S})$ is contained in its basis; R is Cohen-Macaulay if and only if the stable basis equals its basis. (If $\mathscr{S}$ contains any unstable elements then the smallest unstable element in each congruence class will be an element of the unstable basis.) Similar definitions hold for $\mathrm{gr}(\hat{\mathscr{S}})$.

Definition 4.2.9. If $t^x \in \mathrm{gr}(\mathscr{S})$ and $t^x t^d = 0$ then we say that x is an *immediately unstable element of* $\mathrm{gr}(\mathscr{S})$. The analogous definition holds for elements of $\mathrm{gr}(\hat{\mathscr{S}})$.

It follows that the maximum degree of an unstable element is equal to the maximum degree of an immediately unstable element.

Example 4.2.10. We now use Example 4.1.6 to illustrate the above definitions. Recall that $\mathscr{S} = \{1, 7, 9\}$ and $\hat{\mathscr{S}} = \{2, 8, 9\}$. Considering Table 4.3, it is easy to compute the stable and unstable basis of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$. The stable basis of $\mathrm{gr}(\mathscr{S})$ is given by: $\{0, 1, 2, 7, 8, 14, 15, 21, 22\}$ since, for example, $8 \in \mathfrak{M}_2$ is stable but not equal to $9 + x$ for some $x \in \mathfrak{M}_1$. The unstable basis of $\mathrm{gr}(\mathscr{S})$ is given by: $\{3, 4, 5, 6\}$ since, for example, $3 \in \mathfrak{M}_2$ is unstable but not equal to $9 + x$ for some $x \in \mathfrak{M}_2$. Similarly, the stable basis of $\mathrm{gr}(\hat{\mathscr{S}})$ is given by: $\{0, 2, 8, 10, 16, 24, 32, 40, 48\}$, whereas the unstable basis of $\mathrm{gr}(\hat{\mathscr{S}})$ is given by: $\{4, 6, 12, 14\}$. One can also use the description of the sets $\mathfrak{M}_i$ and $\hat{\mathfrak{M}}_i$ in Example 4.1.6 and Proposition 4.2.4 to verify that the basis $\mathcal{B}$ of Example 4.1.6 is as claimed.

We are able to give an explicit bound on the degree of stable basis elements.

Proposition 4.2.11. *Let $\mathscr{S} = \{m_1, \ldots, m_n\}$ be defined as above, let t^x be a stable basis element of $\mathrm{gr}(\mathscr{S})$ and let s^y be a stable basis element of $\mathrm{gr}(\hat{\mathscr{S}})$. Then $\mathrm{ord}_{\mathscr{S}}(x) \leq d - n + 1$, $\mathrm{ord}_{\hat{\mathscr{S}}}(y) \leq d - n + 1$, $\deg(t^x) \leq d - n + 1$ and $\deg(s^y) \leq d - n + 1$.*

Proof. Let J and $\hat{J}$ be the ideals of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ consisting of their unstable elements. Then $\overline{t^d}$ and $\overline{s^d}$ are non-zero divisors in degree one of $\mathrm{gr}(\mathscr{S})/J$ and $\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}$ respectively. Thus, $\Delta H_{\mathrm{gr}(\mathscr{S})/J}(i) = H_{(\mathrm{gr}(\mathscr{S})/J)/\overline{t^d}}(i)$ and $\Delta H_{\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}}(i) = H_{(\mathrm{gr}(\hat{\mathscr{S}})/\hat{J})/\overline{s^d}}(i)$. Since $\mathrm{gr}(\mathscr{S})/J$ and $\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}$ are 1-dimensional and standard graded, $(\mathrm{gr}(\mathscr{S})/J)/\overline{t^d}$ and $(\mathrm{gr}(\hat{\mathscr{S}})/\hat{J})/\overline{s^d}$ are 0-dimensional and standard graded. Thus, the Hilbert functions $H_{(\mathrm{gr}(\mathscr{S})/J)/\overline{t^d}}(i)$ and $H_{(\mathrm{gr}(\hat{\mathscr{S}})/\hat{J})/\overline{s^d}}(i)$ are non-zero until reaching constant value zero so that the Hilbert functions $H_{\mathrm{gr}(\mathscr{S})/J}(i)$ and $H_{\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}}(i)$ strictly increase and then become constant. In some degree all stable basis elements will have appeared, either in that degree or some lower degree. The Hilbert functions, $H_{\mathrm{gr}(\mathscr{S})/J}(i)$ and $H_{\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}}(i)$, become constant precisely when we have them all. The first representative of a congruence class in a graded piece is a stable basis element. Hence, increasing the degree by one yields at least one new congruence class so that in degree $d-n+1$ all stable basis elements have appeared so that the result follows. (We have n congruence class in degree 1 so in degree $d-n+1$ we have all d congruence classes.) $\square$

In what follows, we sometimes refer to the Hilbert functions of $\mathrm{gr}(\mathscr{S})/J$, $\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}$ as the stable Hilbert functions of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively.

Example 4.2.12. The Hilbert functions of $\mathrm{gr}(\mathscr{S})/J$, $\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}$ need not reach a constant value d at the same time. For example, if $\mathscr{S} = \{1,5,7\}$, then $\hat{\mathscr{S}} = \{2,6,7\}$, and the stable Hilbert functions are described in Table 4.2.12.

i	$\mathfrak{M}_i$ mod unstable elements	$H_{\mathrm{gr}(\mathscr{S})/J}(i)$	$\hat{\mathfrak{M}}_i$ mod unstable elements	$H_{\mathrm{gr}(\hat{\mathscr{S}})/\hat{J}}(i)$
1	{1,5,7}	3	{2,6,7}	3
2	{2,6,8,10,12,14}	6	{8,9,12,13,14}	5
3	{9,11,13,15,17,19,21}	7	{15,16,18,19,20,21}	6
4	{16,18,20,22,24,26,28}	7	{22,23,24, 25,26,27,28}	7

Table 4.5: The data associated to Example 4.2.12

We now relate the degree of elements of $\tilde{S}\backslash S$ and the degree of immediately unstable elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$.

Lemma 4.2.13. *The following are equivalent for a fixed $i \geq 0$.*

1. *There exists $a = (a_1, a_2) \in (\tilde{S}\backslash S)_i$.*

2. *$t^{a_2} \in \mathrm{gr}(\mathscr{S})_{i+1}$ and t^{a_2} is an immediately unstable element of $\mathrm{gr}(\mathscr{S})$.*

3. $s^{a_1} \in \mathrm{gr}(\hat{\mathscr{S}})_{i+1}$ *and* s^{a_1} *is an immediately unstable element of* $\mathrm{gr}(\hat{\mathscr{S}})$.

Moreover, the maximum degree of an unstable element of $\mathrm{gr}(\mathscr{S})$ *or* $\mathrm{gr}(\hat{\mathscr{S}})$ *is equal to* $\max\{\deg(x) \mid x \in \tilde{S}\backslash S\} + 1$.

Proof. We first show the equivalence of statement 1 and statement 2. The equivalence of statement 1 and statement 3 follows similarly whence statement 2 is equivalent to statement 3. Recall that for $j \geq 0$, the set $A_j = \{x \in S_j \mid x - (d, 0) \notin S\}$ corresponds to a $\mathbb{K}$-basis of $\mathrm{gr}(\mathscr{S})_j$. Let $t^{a_2} \in \mathrm{gr}(\mathscr{S})_{i+1}$ and suppose $t^{a_2}t^d = 0$ in $\mathrm{gr}(\mathscr{S})$. This is true if and only if $\mathrm{ord}_{\mathscr{S}}(a_2 + d) \leq \mathrm{ord}_{\mathscr{S}}(a_2) = i + 1$, equivalently, if and only if there exists $(y, a_2) \in A_{i+1}$ such that $(y, a_2 + d) \in S_{i+2}\backslash A_{i+2}$; if and only if $(y, a_2) \in S_{i+1}$, $(y - d, a_2) \in (G(S)\backslash S)_i$ and $(y - d, a_2 + d) \in S_{i+1}$; if and only if $(y - d, a_2) \in (\tilde{S}\backslash S)_i$. Thus, statement 1 is equivalent to statement 3. Since the maximum degree of an unstable element is equal to the maximum degree of an immediately unstable element, the last assertion follows. □

Proposition 4.2.14. *Let i be the smallest integer such that $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ is an isomorphism for all $j \geq i$. Let l be the smallest integer such that $s^d : \mathrm{gr}(\hat{\mathscr{S}})_j \to \mathrm{gr}(\hat{\mathscr{S}})_{j+1}$ is an isomorphism for all $j \geq l$. Then $i = l$.*

Proof. By assumption, all unstable elements of $\mathrm{gr}(\mathscr{S})$ have degree $\leq i - 1$. Lemma 4.2.13 implies that the same is true for $\mathrm{gr}(\hat{\mathscr{S}})$. Thus, s^d is injective for all $j \geq i$. To see that it is surjective, by assumption, $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ is an isomorphism for all $j \geq i$. Thus, by Corollary 4.2.2, $s^d : \mathrm{gr}(\hat{\mathscr{S}})_j \to \mathrm{gr}(\hat{\mathscr{S}})_{j+1}$ is onto for all $j \geq i$. □

Proposition 4.2.14 motivates the following definition.

Definition 4.2.15. Let $i \geq 0$ be the smallest integer such that $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ is an isomorphism for all $j \geq i$. (By Proposition 4.2.14 this is equivalent to saying the same thing for s^d and $\mathrm{gr}(\hat{\mathscr{S}})$.) Then we say that S has *stabilized* in degree i.

We now use Definition 4.2.15 to partition the elements $S \in \mathscr{C}'$, constructed from $\mathscr{S}$ as above, into three classes.

Definition 4.2.16. Suppose that S has stabilized in degree i. If $t^d : \mathrm{gr}(\mathscr{S})_{i-1} \to \mathrm{gr}(\mathscr{S})_i$ is not onto, then we say that S is a *case 1 curve.*

If $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j = i-1$ then we say that S is a *case 2 curve.*

If $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for some $j < i-1$, then we say that S is a *case 3 curve.*

Proposition 4.2.14 implies that $S_{\mathscr{S}}$ and $S_{\hat{\mathscr{S}}}$ are of the same case. It is obvious that these cases partition $\mathscr{C}'$. Moreover, we could continue with arbitrarily many cases although, for the moment, this seems to be the right way to partition $\mathscr{C}'$ as we will see in Theorems 4.2.17 and 5.9.1. Before continuing with examples, we present the main result of this section which gives a characterization of the cases.

Theorem 4.2.17. *Let $S \in \mathscr{C}'$ so that S has minimal generating set constructed from $\mathscr{S} = \{m_1, \ldots, m_n\}$, such that $\gcd(\{m_i\}) = 1$ as above.*

1. *The set $\mathscr{S}$ defines a case 1 curve which stabilizes in degree i if and only if there exists a stable basis element of $\mathrm{gr}(\mathscr{S})$ of degree i, all basis elements of $\mathrm{gr}(\mathscr{S})$ have degree $\leq i$ and all unstable elements have degree $\leq i-1$. The same is true for $\mathrm{gr}(\hat{\mathscr{S}})$.*

2. *The set $\mathscr{S}$ defines a case 2 curve which stabilizes in degree i if and only if there exists a basis element of $\mathrm{gr}(\mathscr{S})$ of degree $i-1$, all basis elements have degree*

$\leq i-1$, both $\mathscr{S}$ and $\hat{\mathscr{S}}$ contain an unstable element of degree $i-1$, and all of their unstable elements have degree $\leq i-1$.

3. *The set $\mathscr{S}$ defines a case 3 curve which stabilizes in degree i if and only if both $\mathscr{S}$ and $\hat{\mathscr{S}}$ contain an unstable element of degree $i-1$, all of their unstable elements have degree $\leq i-1$ and all of the basis elements of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ have degree $< i-1$.*

In all cases, suppose S has stabilized in degree i and consider the ideals

$$J = \langle t^x \mid x \text{ is an unstable element of } \mathscr{S} \rangle,$$

$$\hat{J} = \langle s^x \mid x \text{ is an unstable element of } \hat{\mathscr{S}} \rangle.$$

Then

$$H_{\operatorname{gr}(\mathscr{S})}(j) = H_{\operatorname{gr}(\mathscr{S})/J}(j) = H_{\operatorname{gr}(\hat{\mathscr{S}})}(j) = H_{\operatorname{gr}(\hat{\mathscr{S}})/\hat{J}}(j) = d$$

for all $j \geq i$.

Proof. We prove the forward direction for each of the three characterizations. For each characterization the converse follows immediately from the definitions. The last statement is immediate since in all situations there are no unstable elements in degrees $\geq i$. In all cases, by assumption, i is the smallest integer for which $t^d : \operatorname{gr}(\mathscr{S})_j \to \operatorname{gr}(\mathscr{S})_{j+1}$ and $s^d : \operatorname{gr}(\hat{\mathscr{S}})_j \to \operatorname{gr}(\hat{\mathscr{S}})_{j+1}$ are isomorphisms for all $j \geq i$. Thus, by Corollary 4.2.2, all basis elements of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ have degree $\leq i$ and, by Lemma 4.2.13, all unstable elements of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ have degree $\leq i-1$. If $\mathscr{S}$ defines a case 1 curve then, by definition, $t^d : \operatorname{gr}(\mathscr{S})_{i-1} \to \operatorname{gr}(\mathscr{S})_i$ is not onto so, by Theorem 4.2.1, both $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ contain basis elements in degree i. Thus, all basis elements of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ in degree i are stable. If $\mathscr{S}$ defines a case 2 curve, then by definition, $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j = i-1$. Thus, by Theorem 4.2.1, both $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ contain basis elements in degree $i-1$ and, by Corollary 4.2.2, all of their basis elements have degree $\leq i-1$. Since S has stabilized in degree i, $t^d : \mathrm{gr}(\mathscr{S})_{i-1} \to \mathrm{gr}(\mathscr{S})_i$ and $s^d : \mathrm{gr}(\hat{\mathscr{S}})_{i-1} \to \mathrm{gr}(\mathscr{S})_i$ are not injective so Lemma 4.2.13 implies that both $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ have an immediately unstable element of degree $i-1$. Finally, if $\mathscr{S}$ defines a case 3 curve then, by definition, $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j < i-1$ so Corollary 4.2.2 implies that all basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ have degree $< i-1$. Since S has stabilized in degree i, Lemma 4.2.13 implies that both $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ have an immediately unstable element of degree $i-1$. □

Examples 4.2.18.

- If $\mathscr{S}$ is Cohen-Macaulay, then we are always in case 1.
- The curve $\mathscr{S} = \{2, 12, 15\}$ is an example of case 3. It stabilizes in degree 8. It is not case 1 since the stable Hilbert function (i.e., the Hilbert function of $\mathrm{gr}(\mathscr{S})/J$) is $\{1, 3, 6, 9, 12, 14, 15, 15, 15, \to\}$, so the maximum degree of a stable basis element is 6, whereas the Hilbert function (of $\mathrm{gr}(\mathscr{S})$) is $\{1, 3, 6, 10, 15, 18, 18, 16, 15, \to\}$, which shows that there are unstable elements in degree 7. It is not case 2 since the maximum degree of a basis element is 6 corresponding to the basis elements $\{\{38, 52\}, \{68, 22\}\}$. There is an unstable element of degree 7, corresponding to the element $\{53, 37\}$ of $\tilde{S} \backslash S$ which is of degree 6.
- All of the curves in Example 4.2.3 are case 3.
- The curve $\mathscr{S} = \{5, 9, 11, 20\}$ gives a non-Cohen-Macaulay example of case 1. It stabilizes in degree 5. The maximum degree of a basis element is 5,

corresponding to $\{63, 37\}$ (t^{37} is a stable basis element of $\mathrm{gr}(\mathscr{S})$), whereas the maximum degree of an element of $\tilde{S}\backslash S$ is 3 corresponding to $\{36, 24\}$ and the unstable element 24 of degree 4.

- The curve $\mathscr{S} = \{1, 3, 4\}$ provides an example of case 2. It stabilizes in degree 3. We have that $\{6, 2\}$ is a basis element of S which has degree 2, and 2 is an unstable element of $\mathscr{S}$.

- The curve $\mathscr{S} = \{1, 3, 4, 9, 13\}$ is another non-Cohen-Macaulay example of case 1. It stabilizes in degree 3. The element $\{28, 11\}$ is of degree 3 and corresponds to a stable basis element of $\mathrm{gr}(\mathscr{S})$, whereas the maximum degree of an unstable element is 2, corresponding to 5 or the element $\{8, 5\}$ of $\tilde{S}\backslash S$ which is of degree 1. The Hilbert function is given by $\{1, 5, 13, 13, 13, \rightarrow\}$, whereas the stable Hilbert function is given by $\{1, 5, 12, 13, 13\}$.

- We now provide some case 3 curves which suggest that it may be appropriate to continue with arbitrarily many cases. If $\mathscr{S} = \{5, 9, 13, 17, 33, 45, 101, 125\}$, then $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j = 6$, but does not stabilize until degree 9. If $\mathscr{S} = \{5, 9, 13, 17, 33, 45, 301, 325\}$ then $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j = 13$, but does not stabilize until degree 17. If $\mathscr{S} = \{5, 9, 13, 17, 33, 45, 901, 925\}$ then $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ first becomes onto for $j = 38$, but does not stabilize until degree 42.

4.3 A bound on minimal generators

Recall that S is minimally generated by $\Lambda = \{\mathbf{a}_0 = (d, 0), \mathbf{a}_1 = (d - m_1, m_1), \ldots, \mathbf{a}_n = (0, d)\}$ and that $R = \mathbb{K}[s^d, s^{d-m_1}t^{m_1}, \ldots, s^{d-m_{n-1}}t^{m_{n-1}}, t^d]$. Recall also that $R \cong B/\mathfrak{p}$, where $B = \mathbb{K}[X_0, \ldots, X_n]$ is S-graded, setting $\deg(X_i) = \mathbf{a}_i$, and $\mathfrak{p}$ is the kernel of the surjective $\mathbb{K}$-algebra homomorphism $B \to R$ sending X_i to $\mathbf{T}^{\mathbf{a}_i}$. Recall that $\Delta_m = \{\sigma \subseteq \{0, \ldots, n\} \mid m - \sum_{i \in \sigma} \mathbf{a}_i \in S\}$. Finally, recall that, by Corollary 3.1.6, $\mathfrak{p}$ has a minimal ideal generator of multidegree m if and only if Δ_m is disconnected. We now use the theory of the previous section to give a direct proof which bounds the degree of minimal ideal generators of $\mathfrak{p}$.

Lemma 4.3.1. *Suppose that $t^d : \mathrm{gr}(\mathscr{S})_k \to \mathrm{gr}(\mathscr{S})_{k+1}$ is onto, let $j \geq k+2$ and let $m \in A_j$, $m' \in C_j$. Then Δ_m and $\Delta_{m'}$ are connected.*

Proof. Let $m \in A_j$. Since $t^d : \mathrm{gr}(\mathscr{S})_k \to \mathrm{gr}(\mathscr{S})_{k+1}$ is onto and since $j \geq k+2$, the equivalence of statements 1 and 4 of Corollary 4.2.2 implies that $m = (0, d) + a$ for some $a \in A_{j-1}$ so that $\{n\} \in \Delta_m$. Now let $\{v\} \neq \{n\}$ be a vertex of Δ_m. Considering Definition 2.3.8, we may claim that $\{v, n\}$ is a face of Δ_m. To have $\{v\} \in \Delta_m$ implies that $m - \mathbf{a}_v \in S$, so that $m = \mathbf{a}_v + c$ for some $c \in S_{j-1}$. Since $j - 1 \geq k + 1$ we have that $c = c' + (0, d)$, for some $c' \in S_{j-2}$, since otherwise, again by the equivalence of statements 1 and 4 of Corollary 4.2.2, $m = \mathbf{a}_v + c' + (d, 0)$ so that $m \notin A_j$ a contradiction. Thus $m - \mathbf{a}_v - (0, d) = c' \in S$ so that $\{v, n\} \in \Delta_m$. Transposing the argument shows that $\Delta_{m'}$ is also connected. □

Lemma 4.3.2. *Let i be the integer for which S has stabilized, let $j \geq i+1$ and let $\phi : S_{j-2} \to S_j \backslash (A_j \cup C_j)$ defined by $m \mapsto m + (d, d)$. Then ϕ is a bijection.*

Proof. Since S has stabilized in degree i and since $j \geq i + 1$ by last statement of Theorem 4.2.17, $\Delta H_R(j-1) = H_{\mathrm{gr}(\mathscr{S})}(j-1) = d$ and $\Delta H_R(k) = H_{\mathrm{gr}(\mathscr{S})}(k) = d$ for all $k \geq j - 1$. Since $\Delta H_R(j) = H_R(j) - H_R(j-1)$, we have that $H_R(j) = d + H_R(j-1) = d + H_R(j-2) + \Delta H_R(j-1) = 2d + H_R(j-2)$. In terms of S, we have

$|S_j| = 2d + |S_{j-2}|$ and $|A_j| = |C_j| = d$. Again, since $j \geq i+1$, $t^d : \mathrm{gr}(\mathscr{S})_{j-1} \to \mathrm{gr}(\mathscr{S})_j$ is onto so that by the equivalence of statements 1 and 5 of Corollary 4.2.2, $A_j \cap C_j = \emptyset$. Thus, $|S_j \backslash (A_j \cup C_j)| = |S_{j-2}|$. Since S is cancellative, $S_{j-2} + (d, d)$ produces $|S_{j-2}|$ distinct elements of S_j. Thus, ϕ is a bijection. $\square$

Lemma 4.3.3. *Let i be the integer for which S has stabilized. Suppose S is a case 1 curve, let $j \geq i+2$ and let $m \in S_j \backslash (A_j \cup C_j)$. Then Δ_m is connected. Suppose S is a case 2 or a case 3 curve. Let $j \geq i+1$, and let $m \in S_j \backslash (A_j \cup C_j)$. Then Δ_m is connected.*

Proof. In all three cases $j \geq i+1$ so that if $m \in S_j \backslash (A_j \cup C_j)$ then, by Lemma 4.3.2, $m = y + (d, d)$ for some $y \in S_{j-2}$ so that $\{0, n\}$ is a face of Δ_m. Now consider an arbitrary vertex $\{v\} \in \Delta_m$, $v \neq n$, $v \neq 0$. This implies that $m = \mathbf{a}_v + y$ for some $y \in S_{j-1}$. If S is a case 1 curve then, since $j \geq i+2$ by assumption, $t^d : \mathrm{gr}(\mathscr{S})_{j-2} \to \mathrm{gr}(\mathscr{S})_{j-1}$ is onto so that, by the equivalence of statements 1 and 4 of Corollary 4.2.2, $y - (0, d) \in S$ or $y - (0, d) \in S$. Thus, either $\{v, n\} \in \Delta_m$ or $\{0, v\} \in \Delta_m$. If S is a case 2 or a case 3 curve then, since $j \geq i+1$ by assumption, $t^d : \mathrm{gr}(\mathscr{S})_{j-2} \to \mathrm{gr}(\mathscr{S})_{j-1}$ is onto so that again, by the equivalence of statements 1 and 4 of Corollary 4.2.2, $\{v, n\}$ or $\{0, v\}$ is a face of Δ_m. In all three cases, considering Definition 2.3.8, Δ_m is connected. $\square$

We now come to the main result of this section.

Theorem 4.3.4. *Let i be the integer for which S has stabilized. Suppose that S is a case 1 curve. Let $j \geq i+2$ and let $m \in S_j$. Then Δ_m is connected. Suppose that S is a case 2 or 3 curve. Let $j \geq i+1$ and let $m \in S_j$. Then Δ_m is connected. If S is a case 1 curve then all minimal ideal generators of $\mathfrak{p}$ have degree (in the standard*

$\mathbb{N}$-grading) less than or equal to $i+1$. If S is a case 2 or 3 curve all minimal ideal generators of $\mathfrak{p}$ have degree (in the standard $\mathbb{N}$-grading) less than or equal to i.

Proof. Suppose S is a case 1 curve. Then, by definition, $t^d : \mathrm{gr}(\mathscr{S})_{i-1} \to \mathrm{gr}(\mathscr{S})_i$ is not onto but $t^d : \mathrm{gr}(\mathscr{S})_i \to \mathrm{gr}(\mathscr{S})_{i+1}$ is onto. Thus, if $j \geq i+2$ and $m \in A_j$ or C_j then Lemma 4.3.1 implies that Δ_m is connected. On the other hand, if $j \geq i+2$ and $m \in S_j \backslash (A_j \cup C_j)$ then Lemma 4.3.3 implies that Δ_m is connected. If S is a case 2 or case 3 curve then, by definition, $t^d : \mathrm{gr}(\mathscr{S})_{i-1} \to \mathrm{gr}(\mathscr{S})_i$ is onto. Thus, if $j \geq i+1$ and $m \in A_j$ or C_j then Lemma 4.3.1 implies that Δ_m is connected. On the other hand, if $j \geq i+1$ and $m \in S_j \backslash (A_j \cup C_j)$ then Lemma 4.3.3 implies that Δ_m is connected. $\square$

Examples 4.3.5. If $\mathscr{S} = \{1, 2, 3\}$ then S is a case 1 curve which stabilizes in degree 1 and $\mathfrak{p}$ is minimally generated by binomials of degree 2. Thus, Theorem 4.3.4 is sharp for case 1 curves. If $\mathscr{S} = \{1, 3, 4\}$ then S is a case 2 curve which stabilizes in degree 3 and $\mathfrak{p}$ is minimally generated by a binomial of degree 2 and 3 binomials of degree 3. Thus, Theorem 4.3.4 is sharp for case 2 curves. I have yet to find an example for which Theorem 4.3.4 is sharp for case 3 curves.

4.4 Decomposition via congruence classes

Proposition 4.2.11 and Theorem 4.2.17 imply that if S is a case 1 curve then the integer for which S stabilizes is less than or equal to $d-n+1$. We would like to obtain a similar result for case 2 and case 3 curves. For this, we further develop an approach to study $R = \mathbb{K}[S]$ which is used in [29] and considered in higher dimensions by Ping Li and Leslie Roberts. The right context in which to view the following discussion is in the context of monomial modules.

Definition 4.4.1. Let S be a subsemigroup of $\mathbb{N}^2_d = \{(x, y) \in \mathbb{N}^2 \mid x + y \equiv 0 \bmod d\}$ minimally generated by $(d, 0), (0, d)$ and some other generators, and let $R = \mathbb{K}[S] \subseteq \mathbb{K}[s, t]$. A *monomial* $\mathbb{K}[s^d, t^d]$*-module* is a submodule of $R(-c)$, $c \in \mathbb{Z}$, which has a finite generating set $\{\mathbf{T}^b = s^{b_1}t^{b_2} \mid b = (b_1, b_2) \in S\} \subseteq R(-c)$ of monomials.

The advantage of monomial modules is that given monomial modules M and N, $N \subseteq M$, we can form "staircase diagrams", as we will see in Figure 4.2, which represent the monomials of M and the monomials of M/N.

Example 4.4.2. If $S \in \mathscr{C}'$ then $R = \mathbb{K}[S]$ is a monomial $\mathbb{K}[s^d, t^d]$-module with generating set $\{\mathbf{T}^b \mid b \in \mathcal{B}\}$. We will see several other examples shortly.

From now on we assume that $S \in \mathscr{C}'$ and all other notation as before. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$. We say that $\alpha \equiv \beta \bmod d$ if $\alpha_i - \beta_i \equiv 0 \bmod d$, $1 \leq i \leq 2$. Since the collection of elements of $\mathscr{S}$ are relatively prime, in $G(S)$ there are d congruence classes $\bmod\, d$ with representatives $\mathcal{C} = \{(0, 0)\} \cup \{\alpha = (d - \alpha_2, \alpha_2) \mid 0 < \alpha_2 \leq d - 1\}$. For each $\alpha \in \mathcal{C}$ let $S'_{<\alpha>} = \{a \in S' \mid a \equiv \alpha \bmod d\}$, let $S_{<\alpha>} = \{a \in S \mid a \equiv \alpha \bmod d\}$ and let $\mathcal{B}_{<\alpha>} = \{a \in \mathcal{B} \mid a \equiv \alpha \bmod d\}$. By construction, $S_{<\alpha>} \subseteq S'_{<\alpha>}$. Let $\mathbb{K}[S']_{<\alpha>} = \oplus_{a \in S'_{<\alpha>}} \mathbb{K}[S']_a$ and let $R_{<\alpha>} = \oplus_{a \in S_{<\alpha>}} R_a$. We use $< \alpha >$, as opposed to α, in order to distinguish between congruence classes $R_{<\alpha>}, \alpha \in \mathcal{C}$, and graded pieces R_a, $a \in S$. Congruence class $\bmod\, d$ induces decompositions $S' = \coprod_{\alpha \in \mathcal{C}} S'_{<\alpha>}$, $S = \coprod_{\alpha \in \mathcal{C}} S_{<\alpha>}$, $\mathbb{K}[S'] = \oplus_{\alpha \in \mathcal{C}} \mathbb{K}[S']_{<\alpha>}$ and $R = \oplus_{\alpha \in \mathcal{C}} R_{<\alpha>}$. Let $x = s^d$, $y = t^d$. Regarding $\mathbb{K}[x, y]$ as a subalgebra of R, $x \mapsto s^d, y \mapsto t^d$, by construction, each $R_{<\alpha>}$ is a *monomial* $\mathbb{K}[x, y]$*-module* with minimal generating set consisting of those monomials $\mathbf{T}^b = s^{b_1}t^{b_2}$ such that $b = (b_1, b_2) \in \mathcal{B}$ and $b \equiv \alpha \bmod d$, or equivalently,

$\{\mathbf{T}^b \mid b \in \mathcal{B}_{<\alpha>}\}$. Moreover, $R_{<\alpha>}$ is a submodule of the $\mathbb{K}[x,y]$-module $\mathbb{K}[S']_{<\alpha>}$. We will see momentarily that $\mathbb{K}[S']_{<\alpha>}$ is also a monomial module.

For a set $A \subseteq \mathbb{N}^2$, let x_1 and x_2 denote the smallest first and second coordinates of all elements of A, respectively, and define $\inf A := (x_1, x_2)$. Let $b_\alpha = (b_1, b_2) = \inf S'_{<\alpha>}$. We have the following lemma.

Lemma 4.4.3. *Let $b_\alpha = \inf S'_{<\alpha>}$. Then $b_\alpha \in S'_{<\alpha>}$ and $b_\alpha = \inf S_{<\alpha>} = \inf \mathcal{B}_{<\alpha>}$. Moreover, $b_\alpha \in \mathcal{B}_{<\alpha>}$ if and only if $\mathcal{B}_{<\alpha>}$ contains one element if and only if $S'_{<\alpha>} = S_{<\alpha>}$.*

Proof. Let $b_\alpha = (b_1, b_2) = \inf S'_{<\alpha>}$. Then there exists $x = (b_1, x_2) \in S'_{<\alpha>}$. If $b_2 \neq x_2$ then $b_2 < x_2$. Since $b_2 \equiv x_2 \bmod d$, $x_2 = qd + b_2$, $q > 0$. Thus, $b_\alpha + q(0,d) = x \in S'_{<\alpha>}$ which also implies that $b_\alpha \in S'_{<\alpha>}$. Let $a_\alpha = (a_1, a_2) = \inf \mathcal{B}_{<\alpha>}$. By definition of $\mathcal{B}$, $a_\alpha = \inf S_{<\alpha>}$. Obviously, $a_1 \geq b_1$ and $a_2 \geq b_2$. Since $b_\alpha \in S'_{<\alpha>}$, we have that $b_\alpha + p(d,0) \in S$ and $b_\alpha + m(0,d) \in S$ for some $p, m \geq 0$. Take the smallest such m. We claim that $(b_1, b_2 + md) \in \mathcal{B}_{<\alpha>}$. We have $(b_1, b_2 + md) \in S_{<\alpha>}$ but $(b_1, b_2 + md) - (0,d) \notin S_{<\alpha>}$. Since $(b_1, b_2) = \inf S'_{<\alpha>}$, we have $(b_1, b_2 + md) - (d, 0) \notin S'_{<\alpha>}$, thus, $(b_1, b_2 + md) \notin S_{<\alpha>}$. Thus, $(b_1, b_2 + md) \in \mathcal{B}_{<\alpha>}$ so that $a_1 \leq b_1$. A similar argument shows $a_2 \leq b_2$. Thus, $a_1 = b_1$ and $a_2 = b_2$. For the last assertion it is immediate from the definitions that $S'_{<\alpha>} = S_{<\alpha>}$ if and only if $\mathcal{B}_{<\alpha>}$ contains only one element. Since $b_\alpha \in S'_{<\alpha>}$, $b_\alpha \in \mathcal{B}_{<\alpha>}$ if and only if $S'_{<\alpha>} = S_{<\alpha>}$. $\square$

Let $b_\alpha = \inf S'_{<\alpha>}$. Then Lemma 4.4.3 implies that a minimal generating set for $\mathbb{K}[S']_{<\alpha>}$ as a $\mathbb{K}[x,y]$-module is the monomial $\mathbf{T}^{b_\alpha}$, so that $\mathbb{K}[S']_{<\alpha>}$ is also a monomial module. There is a bijective correspondence between elements of $S'_{<\alpha>}$ and monomials of $\mathbb{K}[x,y]$ given by $(a_1, a_2) \mapsto \mathbf{x}^{((a_1-b_1)/d, (a_2-b_2)/d)} = x^{(a_1-b_1)/d} y^{(a_2-b_2)/d}$. Letting $b'_\alpha = \deg(b_\alpha) = (b_1 + b_2)/d$, under this identification, an element of $(S'_{<\alpha>})_i$

corresponds to a monomial in $\mathbb{K}[x,y](-b'_\alpha)$ of degree $i - b'_\alpha$. Thus, this identification induces an isomorphism of $\mathbb{Z}$-graded $\mathbb{K}[x,y]$-modules $\mathbb{K}[x,y](-b'_\alpha) \cong \mathbb{K}[S']_{<\alpha>}$ given by $\mathbf{x}^{(a_1,a_2)} \mapsto \mathbf{T}^{(b_1+da_1,b_2+da_2)}$. In a similar manner, the monomials of $R_{<\alpha>}$ are in bijective correspondence with the monomials of a monomial submodule $I_{<\alpha>}$ of $\mathbb{K}[x,y](-b'_\alpha)$.

We now describe a minimal monomial generating set for $I_{<\alpha>}$. By definition of $\mathcal{B}$ if $b, c \in \mathcal{B}_{<\alpha>}$ then b and c must be incomparable with respect to the natural product partial order on $\mathbb{N}^2$. The same is true after subtracting b_α. Thus, the set of monomials $\{\mathbf{x}^{(a_1/d,a_2/d)} \mid a = (a_1,a_2) + b_\alpha \in \mathcal{B}_{<\alpha>}\}$ constitute a minimal generating set for $I_{<\alpha>}$ as a $\mathbb{K}[x,y]$-module. (Note that if $\mathcal{B}_{<\alpha>}$ contains only one element, then $I_{<\alpha>} = \mathbb{K}[x,y](-b'_\alpha)$.) The module $I_{<\alpha>}$ is obviously graded. Moreover, $\mathbf{x}^{(a_1,a_2)} \in I_{<\alpha>}$ if and only if $(da_1+b_1, da_2+b_2) \in S$. Thus, the map $I_{<\alpha>} \to R_{<\alpha>}$, defined by $x^{(a_1,a_2)} \mapsto \mathbf{T}^{(b_1+da_1,b_2+da_2)}$, yields an isomorphism of graded $\mathbb{K}[x,y]$-modules. This map also induces an isomorphism $\oplus_{\alpha\in\mathcal{C}} I_{<\alpha>} \to \oplus_{\alpha\in\mathcal{C}} R_{<\alpha>} \cong R$ of graded $\mathbb{K}[x,y]$-modules.

Suppose now that $\mathcal{B}_{<\alpha>}$ contains more than one element and let $b_\alpha = (b_1,b_2) = \inf S'_{<\alpha>}$. Then, as in the proof of Lemma 4.4.3, there exists elements (b_1, b_2+md), $(b_1+pd, b_2) \in \mathcal{B}_{<\alpha>}$, where $m, p \geq 0$ are the smallest integers such that $b_\alpha + m(0,d)$ and $b_\alpha + p(d,0) \in S$. It follows that $\mathbf{x}^{(0,m)}$ and $\mathbf{x}^{(p,0)}$ will be minimal monomial generators of $I_{<\alpha>}$ and that there are only finitely many monomials not in $I_{<\alpha>}$, i.e., that $\mathbb{K}[x,y](-b_\alpha)/I_{<\alpha>}$ has a finite monomial $\mathbb{K}$-basis. Since the monomials of $\mathbb{K}[x,y](-b'_\alpha)$ are in bijective correspondence with $S'_{<\alpha>}$, this implies that the elements of $S'_{<\alpha>} \backslash S_{<\alpha>}$ are in bijective correspondence with the monomials of $\mathbb{K}[x,y](-b'_\alpha)/I_{<\alpha>}$. We can represent the monomials of $I_{<\alpha>}$ and the monomials not in $I_{<\alpha>}$ by using staircase diagrams (Figure 4.2 for example). Finally, it is also apparent that, the monomial $\mathbf{x}^{(0,m)}$ (of $\mathbb{K}[x,y](-b'_\alpha)/I_{<\alpha>}$) will not be a multiple of x (i.e., a horizontal translate), under the action of $\mathbb{K}[x,y]$ and, similarly, the monomial $\mathbf{x}^{(p,0)}$ will not be a multiple of y (i.e., a vertical translate), under the action of $\mathbb{K}[x,y]$. Converting back into S, it follows that $b_2 + md$ and $b_1 + pd$ will not be the second or first coordinates, respectively of elements of $S' \backslash S$. Thus, $b_2 + md$ will be a stable basis element of $\mathrm{gr}(\mathscr{S})$ and $b_1 + pd$ will be a stable basis element of $\mathrm{gr}(\hat{\mathscr{S}})$.

Since the monomials $\mathbf{x}^{(0,m)}$ and $\mathbf{x}^{(p,0)}$ are minimal $\mathbb{K}[x,y]$-module generators of $I_{<\alpha>}$ and since $b_2 + md$ and $b_1 + pd$ are stable basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively, it follows, from Proposition 4.2.11, that the degree of t^{b_2+md} and s^{b_1+pd} will be less than or equal to $d - n + 1$ and thus, the degrees of $(b_1, b_2 + md)$ and $(b_1 + pd, b_2)$ will be less than or equal to $d - n + 1$. Applying this to $\mathbb{K}[x,y](-b'_\alpha)$, it follows that the monomials $\mathbf{x}^{(0,m)}$ and $\mathbf{x}^{(p,0)}$ have degree $\leq d - n + 1$.

Before continuing with an example, we summarize some of the discussion thus far in the following.

Theorem 4.4.4. *Let $b_\alpha = (b_1, b_2) = \inf S'_{<\alpha>}$ and let $b'_\alpha = \deg(b_\alpha)$. The following hold.*

1. *The semigroup rings $\mathbb{K}[S']$ and $R = \mathbb{K}[S]$ have natural decompositions $\mathbb{K}[S'] = \oplus_{\alpha \in \mathcal{C}} \mathbb{K}[S']_{<\alpha>}$, $R = \oplus_{\alpha \in \mathcal{C}} R_{<\alpha>}$ where α ranges over all congruence classes of $G(S)$.*

2. *The $\mathbb{K}[x,y]$-modules $\mathbb{K}[S']_{<\alpha>}$ and $\mathbb{K}[x,y](-b'_\alpha)$ are isomorphic.*

3. *The elements of $\mathcal{B}_{<\alpha>}$ correspond to minimal monomial $\mathbb{K}[x,y]$-module generators for a monomial module $I_{<\alpha>}$ of $\mathbb{K}[x,y](-b'_\alpha)$. Moreover, the map $\mathbf{x}^{(a_1,a_2)} \mapsto \mathbf{T}^{(b_1+da_1, b_2+da_2)}$ is an isomorphism $I_{<\alpha>} \cong R_{<\alpha>}$ of $\mathbb{Z}$-graded $\mathbb{K}[x,y]$-modules.*

4. *If $\mathcal{B}_{<\alpha>}$ contains only one element then $I_{<\alpha>} = \mathbb{K}[x,y](-b'_\alpha)$ and the monomial $\mathbf{x}^{(0,0)}$ corresponds to the stable basis elements b_2, b_1 of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ in the appropriate congruence class. If $\mathcal{B}_{<\alpha>}$ contains more than one element then there exists minimal monomial generators $\mathbf{x}^{(0,m)}$, $\mathbf{x}^{(p,0)}$, $m, p \geq 0$ of $I_{<\alpha>}$ such that $b_2 + md$ and $b_1 + pd$ are the stable basis elements of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\hat{\mathscr{S}})$ in the appropriate congruence class. Any monomial which corresponds a stable basis element of $\operatorname{gr}(\mathscr{S})$ and $\operatorname{gr}(\mathscr{S})$ has degree $\leq d-n+1$. Moreover, in a given congruence class α there are only finitely many monomials of $\mathbb{K}[x,y](-b'_\alpha)$ which are not in $I_{<\alpha>}$.*

5. *There is a bijective correspondence between a monomial $\mathbb{K}$-basis of $\mathbb{K}[x,y](-b'_\alpha)/I_{<\alpha>}$ and $S'_{<\alpha>} \backslash S_{<\alpha>}$.*

Example 4.4.5. This is a continuation of Example 4.1.6. Recall that $\mathscr{S} = \{1,7,9\}$ so that $\hat{\mathscr{S}} = \{2,8,9\}$ and S is minimally generated by

$$\Lambda = \{\{9,0\},\{8,1\},\{2,7\},\{0,9\}\}.$$

Let $\alpha = \{4,5\}$. Then considering the basis computed in Example 4.1.6, we see that $\mathcal{B}_{<\alpha>} = \{\{4,14\},\{40,5\}\}$ so that $b_\alpha = \inf \mathcal{B}_{<\alpha>} = (4,5)$, $b'_\alpha = 1$. Considering Table 4.3, we see that 14 is a stable basis element of $\operatorname{gr}(\mathscr{S})$, that 5 is an unstable basis element of $\operatorname{gr}(\mathscr{S})$ and that 40 and 4 are stable and unstable basis elements of $\operatorname{gr}(\hat{\mathscr{S}})$ respectively. The staircase diagram of $I_{<\alpha>}$ is shown in Figure 4.2. The elements of $\mathcal{B}_{<\alpha>}$ are represented by large solid dots. The elements of A_j which are not elements of C_j, $0 \leq j \leq 4$, in this congruence class, are represented by the open circles. On the other hand, the elements of C_j which are not elements of A_j, $0 \leq j \leq 6$, in this congruence class, are represented by the open circle with an $\times$ in the middle. There is only one element of $\tilde{S}_{<\alpha>}\backslash S_{<\alpha>}$ in this example; it is represented by a solid black square. The remaining elements of $S'_{<\alpha>}\backslash S_{<\alpha>}$ are represented by open squares. The axis labels indicate the exponents of the corresponding monomials of $\mathbb{K}[x,y](-1)$. (Recall that since $b'_\alpha = 1$ the monomial $\mathbf{x}^{(a_1,a_2)}$ has degree $1+a_1+a_2$.) We also have that $\mathbf{x}^{(0,1)} \mapsto \mathbf{T}^{(4,14)}$ and that $\mathbf{x}^{(4,0)} \mapsto \mathbf{T}^{(40,5)}$. Since $\mathbf{x}^{(0,1)}$ has degree 2 and $\mathbf{x}^{(4,0)}$ has degree 5, the correspondence $I_{<\alpha>} \to R_{<\alpha>}$ is of degree 0 as claimed.

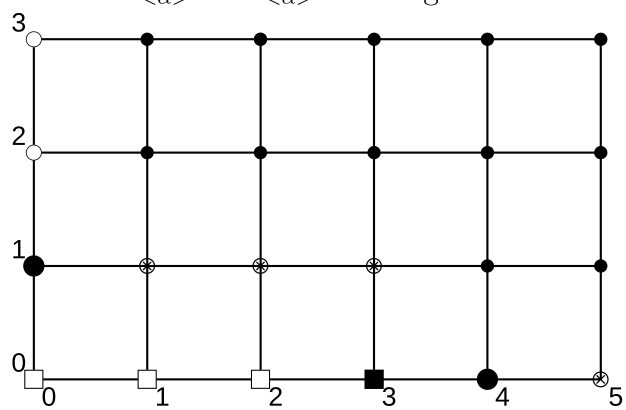


Figure 4.2: The staircase diagram of $I_{<\alpha>=(4,5)}$ for Example 4.4.5.

4.4.1 General bounds for stabilization

Proposition 4.2.11 and Theorem 4.2.17 imply that if S is a case 1 curve then the integer for which S stabilizes is less than or equal to $d-n+1$. We use Theorem 4.4.4 and the discussion of this section to give a similar statement for case 2 and case 3 curves. For this, let $b_\alpha = (b_1,b_2) = \inf S'_{<\alpha>}$, $b'_\alpha = \deg(b_\alpha)$ and, finally, let $(b_1, b_2+md), (b_1+pd, b_2) \in \mathcal{B}_{<\alpha>}$ so that they are the elements of $\mathcal{B}_{<\alpha>}$ corresponding to the stable basis elements in their congruence class of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively. Let $m' = \deg((b_1, b_2+md))$ and let $p' = \deg((b_1+pd, b_2))$. As a consequence of Theorem 4.4.4, for a fixed congruence class, the finite number of monomials which are not in $I_{<\alpha>}$ are contained in a $(p' - b'_\alpha) \times (m' - b'_\alpha)$-rectangle. (If $I_{<\alpha>}$ has more than two generators, some monomials of this rectangle will be contained in $I_{<\alpha>}$.) (If this description is not clear, refer to Figure 4.2 of Example 4.4.5 where $d = 9$, $\alpha = (4, 5)$, $b_\alpha = (4, 5)$, $b'_\alpha = 1$, $(b_1, b_2 + md) = (4, 14)$, $(b_1 + pd, b_2) = (40, 5)$, $m' = 2$, $p' = 5$, and we formed a 4×1 rectangle containing all monomials not in $I_{<\alpha>}$.)

To bound the maximum degree of an element of $S'_{<\alpha>} \backslash S_{<\alpha>}$ we may look at a given rectangle and determine the largest degree of a monomial not in $I_{<\alpha>}$. Moreover, we should make the rectangle as large as possible. This amounts to maximizing p' and m' while minimizing b'_α. By Theorem 4.4.4, m' and p' are less than or equal to $d - n + 1$. Since $b'_\alpha = 0$ if and only if $\alpha = (0, 0)$, in which case $I_{<\alpha>} = \mathbb{K}[x, y]$, we may consider the case $b'_\alpha = 1$. The result is forming a $(d - n) \times (d - n)$ square, Figure 4.3, where the black square and the monomial $x^{d-n-1}y^{d-n-1}$, which is not in $I_{<\alpha>}$,

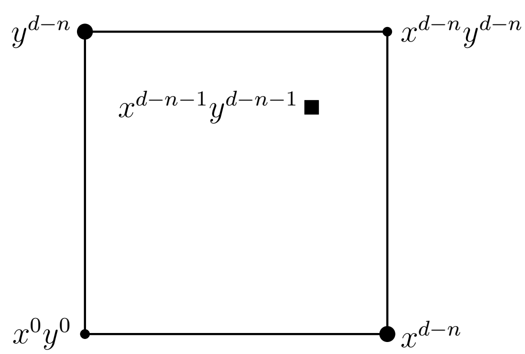


Figure 4.3: An extreme case 2 or case 3 curve.

corresponds to an element of $S'_{<\alpha>} \backslash S_{<\alpha>}$, of maximum degree or equivalently, to an element of $\tilde{S}_{<\alpha>} \backslash S_{<\alpha>}$ of maximum degree. Since $b'_\alpha = 1$, the monomial $x^{d-n-1}y^{d-n-1}$ has degree $2d - 2n - 2 + 1 = 2d - 2n - 1$. This is the maximum degree of an element of $S'_{<\alpha>} \backslash S_{<\alpha>}$ and in fact, the maximum degree of an element of $S' \backslash S$ and $\tilde{S} \backslash S$. Using these observations, we now state the main result of this section.

Theorem 4.4.6. *Let $S \in \mathscr{C}'$ be constructed form $\mathscr{S} = \{m_1, \ldots, m_n = d\}$ as in the previous discussion. The following statements hold.*

1. *If S is a case 2 or case 3 curve then all elements of $S' \backslash S$ have degree less than or equal to $2d - 2n - 1$.*

2. *If S is a case 2 or case 3 curve then the maximum degree of an unstable element of $\mathrm{gr}(\mathscr{S})$ or $\mathrm{gr}(\hat{\mathscr{S}})$ is $\leq 2d - 2n$.*

3. *If S is a case 1 curve then S stabilizes in degree $\leq d - n + 1$. If S is case 2 or case 3 curve then S stabilizes in degree $\leq 2d - 2n + 1$.*

Proof. Statement 1 was proved in the above discussion. Statement 2 follows from Statement 1 and Lemma 4.2.13. The first assertion of Statement 3 follows from Proposition 4.2.11 and Definition 4.2.15. The second assertion follows from Statement 2 and Definition 4.2.15. □

Remark 4.4.7. In Theorem 5.9.1 we will show that for case 2 and case 3 curves the regularity of $\mathfrak{p}$ will be equal the degree for which S stabilizes. By the famous result of [21], $\mathrm{reg}(\mathfrak{p}) \leq d - n + 2$, so that the combinatorial bound of statement 3 of Theorem 4.4.6 is not quite double this bound. We will also show that for case 1 curves the regularity of $\mathfrak{p}$ will be equal to one more than the degree for which S stabilizes. Thus, for case 1 curves we have obtained the appropriate bound. At the moment, for case 2 and 3 curves, we are unable to get the the bound of [21]; the result stated in Theorem 4.4.6 is the best we can do. It is also unclear whether the approach used here can be used to obtain their bound.

Remark 4.4.8. It is also clear from the $(d-n) \times (d-n)$ square formed above that the number of basis elements in a particular congruence class is at most $d - n + 1$, which implies that the number of basis elements of S is at most quadratic in d. Similarly, the number of elements of $S'\backslash S$ in a particular congruence class is at most $(d-n)^2$, so the number of elements of $S'\backslash S$ is at most cubic in d.

Chapter 5

Betti numbers, regularity and stabilization

In this chapter we describe the approach of [14] to determine finite check sets containing all multidegrees of syzygies of $\mathfrak{p}$. This approach is then related to [6] and the theory of Chapter 4. More specifically, we obtain a description for the regularity of $\mathfrak{p}$ in terms of the stabilization of S (Definition 4.2.15). We then give insight into how the case of S (Definition 4.2.16 and Theorem 4.2.17) is reflected by how the regularity is obtained.

5.1 Introduction

We keep the same assumptions on the semigroup S as in Chapter 4, which we will review shortly. In the mean time, recall that $\dim_{\mathbb{K}} \tilde{H}_t(\Delta_m) = \beta_{t,m}, t \geq 0$, where $m \in S$, $\beta_{t,m} = \dim_{\mathbb{K}} \operatorname{Tor}_t^B(\mathbb{K}, \mathfrak{p})_m$ and $\mathfrak{p}$ is the kernel, of the $\mathbb{K}$-algebra homomorphism $B = \mathbb{K}[X_0, \ldots, X_n] \to R = \mathbb{K}[S]$, defined by sending $X_i \mapsto \mathbf{T}^{\mathbf{a}_i}$. We would like to find all nontrivial Betti numbers (or a least a finite set of multidegrees from which we can compute them). At present our goal is as follows: Give a sufficient condition for $\tilde{H}_t(\Delta_m) = 0$ for some $t \geq 0$ and for some $m \in S$. In fact, we are hoping to do better. We would like to find all $m \in S$ such that $\tilde{H}_t(\Delta_m) \neq 0$ for some $t \geq 0$.

For now, we ignore the graphs and their homology defined in [14]. In that account, the homology of each graph is isomorphic to the homology of an appropriate reduced (perhaps relative) chain complex of appropriate simplicial complexes. The motivation for their graphs seem to be to compute a $\mathbb{K}$-basis for $\tilde{H}_t(\Delta_m)$, $t \geq -1$. At present we already know how to do this (at least with our current assumptions on S).

We now introduce some notation. We try to be consistent, to some extent, with that of [14] so as to facilitate verifying this write up with theirs while at the same time being consistent with the notation already introduced in this thesis.

Let $\Lambda = E \cup A$, where $E = \{\mathbf{a}_0 = e = (d, 0), \mathbf{a}_n = e' = (0, d)\}$ and $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$, be a minimal generating set for $S \subseteq \mathbb{N}^2$ constructed from $\mathscr{S} = \{m_1, \ldots, m_n = d\}$ by setting $\mathbf{a}_i = (d - m_i, m_i), 1 \leq i \leq n$. (Note that $|\Lambda| = n + 1$, and that $|A| = n - 1$.) Let $F \subseteq \Lambda$, and let $n_F := \sum_{n \in F} n$. (If $F = \emptyset$, then $n_F = 0$.) Let $m \in S$ and define Δ_m to be the simplicial complex on Λ defined as follows:

$$\Delta_m := \{F \subseteq \Lambda \mid m - n_F \in S\}.$$

This definition agrees with Definition 3.1.1 although in what follows, for notational reasons, it is convenient to use subsets of Λ as the faces of Δ_m as opposed to subsets of the indices. This being said, in all examples where we actually compute Δ_m and any other simplicial complexes, as in the previous Chapters, we will label the vertices from $1, \ldots, n + 1$, so that the vertex corresponding to $\mathbf{a}_i$ will be $i + 1$.

If we are defining a simplicial complex by facets, we explicitly state that we are doing so (as opposed to setting it equal to its set of facets). This is to avoid confusion with some of the definitions of [14] which define some simplicial complexes by setting them equal to all of their faces.

So as to have a concrete choice of boundary maps in $\tilde{C}.(\Delta)$, we must define a total ordering on Λ. Without loss of generality, assume this is $e < \mathbf{a}_1 < \cdots < \mathbf{a}_{n-1} < e'$. Now, with respect to this total ordering, the boundary maps of $\tilde{C}.(\Delta)$ are the same as if we were defining Δ_m to be a simplicial complex with vertex set $\{0, \ldots, n\}$.

Throughout we use the convention that $A \backslash B$ denotes the set theoretic complement of sets A and B. In [14] the authours use $A - B$, and we would like to avoid confusion with other accounts of affine semigroups. More specifically, in [32] $A - B$ means something different.

5.2 The idea

We now give a brief summary of the approach used in this chapter. In [14] a simplicial subcomplex K_m, of Δ_m is defined. The relative homology $\tilde{H}_t(\Delta_m, K_m)$ of the relative chain complex $\tilde{C}.(\Delta_m, K_m)$ is then considered. Recall that this is the quotient of the complexes $\tilde{C}.(\Delta_m)$ and $\tilde{C}.(K_m)$. This leads to the short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}.(K_m) \longrightarrow \tilde{C}.(\Delta_m) \longrightarrow \tilde{C}.(\Delta_m, K_m) \longrightarrow 0\ ,$$

and then a long exact sequence

$$\cdots \longrightarrow \tilde{H}_{t+1}(\Delta_m, K_m) \longrightarrow \tilde{H}_t(K_m) \longrightarrow \tilde{H}_t(\Delta_m) \longrightarrow \tilde{H}_t(\Delta_m, K_m) \longrightarrow \cdots$$

of homology. From this it is clear that if $\tilde{H}_t(\Delta_m) \neq 0$ for some $t \geq 0$ then either $\tilde{H}_t(K_m) \neq 0$ or $\tilde{H}_t(\Delta_m, K_m) \neq 0$. Thus, a sufficient condition for $\tilde{H}_t(\Delta_m) = 0$ is that both $\tilde{H}_t(K_m)$ and $\tilde{H}_t(\Delta_m, K_m)$ equal 0. In Proposition 5.7.2, we give a sufficient condition for $\tilde{H}_t(\Delta_m, K_m) = 0$. In order to study the homology of $\tilde{C}.(K_m)$ an acyclic simplicial complex (i.e., a simplicial complex with trivial homology) $\bar{K}_m$, containing K_m, is defined. This leads to another short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}.(K_m) \longrightarrow \tilde{C}.(\bar{K}_m) \longrightarrow \tilde{C}.(\bar{K}_m, K_m) \longrightarrow 0 .$$

It turns out that K_m and $\bar{K}_m$ can never be equal to $\{\emptyset\}$. Thus, as before, we get a long exact sequence of homology, although this time, since $\tilde{C}.(\bar{K}_m)$ is acyclic, we have that $\tilde{H}_{t+1}(\bar{K}_m, K_m) \cong \tilde{H}_t(K_m)$ for $t \geq -1$. Hence, in order to determine when $\tilde{H}_t(K_m) = 0$, we study $\tilde{H}_{t+1}(\bar{K}_m, K_m)$ and give a sufficient condition for $\tilde{H}_{t+1}(\bar{K}_m, K_m) = 0$ (Proposition 5.7.3).

5.3 Some notation translation

We now summarize the isomorphisms between the "graphic" homology (the homology coming from various graphs) of [14] and the reduced homology from their simplicial complexes.

In [14, Definition 1.1, p. 145], given a subset B of S and a subset A of Λ satisfying certain conditions, a graph $\mathscr{G}_B^A$ is defined. Given an element $m \in G(S)$ a chain complex $\{C.((\mathscr{G}_B^A)_m), \delta.\}$ is then constructed, and its homology $H.((\mathscr{G}_B^A)_m)$ is called the homology of $\mathscr{G}_B^A$ at m. In [14, Section 2, p. 146] the notation for the graphs is simplified. In particular, the graph $\mathscr{G}_B^A$ is denoted simply as $\mathscr{G}_B$, and $\{C.((\mathscr{G}_B^A)_m), \delta.\}$ and $H.((\mathscr{G}_B^A)_m)$ are denoted by $\{C.(B_m), \delta.\}$ and $H.(B_m)$. Also in this section, the subset $A \subseteq \Lambda$ is fixed, and a subset Q of S is defined. Both Q and A are shown to satisfy the conditions of [14, Definition 1.1, p. 145], so that they can consider the graph $\mathscr{G}_Q^A$. Given $m \in G(S)$ along with the previous simplification of the notation, we then have the chain complex $\{C.(Q_m), \delta.\}$ and its homology $H.(Q_m)$. We now be more specific as to how the homology of the graphs translates into homology of simplicial complexes.

- In [14] the set Q, whose definition we will recall in Section 5.4, is equivalent to our basis $\mathcal{B}$. In this chapter we use $\mathcal{B}$ throughout.

- $C.(Q_m) \cong \tilde{C}.(\Delta_m, K_m)$, which implies that $H_t(Q_m) \cong \tilde{H}_t(\Delta_m, K_m)$.

- For all $t \geq -1$, $C_t(\bar{D}_m) \cong \tilde{C}_{t+1}(\bar{K}_m, K_m)$. (Similarly, the symbol $C.(\bar{D}_m)$ is used to denote the chain complex at $m \in S$ which is constructed from the graph $\mathscr{G}_D$. See [14, p. 150].)

Thus, using the above isomorphisms some of the exact sequences etc. of [14], which are written in terms of "graphic" homology, can be made to look like the ones mentioned previously.

5.4 The definitions of K_m and $\bar{K}_m$

We now give the definitions of K_m and $\bar{K}_m$, as presented [14], and make a few comments about them. We also say how some things simplify since we are restricting to two dimensions.

Recall that the set $\mathcal{B}$, [14, Definition 2.1, p. 147] or see Section 4.1.3, denotes the basis. i.e.,

$$\mathcal{B} = \{m \in S \mid m - e \notin S \forall e \in E\},$$

and the set K_m, [14, p. 147] is defined to be:

$$K_m = \{L \in \Delta_m \mid (L \cap E \neq \emptyset) \text{ or } (L \subseteq A \text{ and } m - n_L \in S\backslash\mathcal{B})\}.$$

We show now that K_m is a simplicial subcomplex of Δ_m. As a set it is clear that $K_m \subseteq \Delta_m$. To see that K_m is closed under taking subsets, the only nontrivial case to check is if $L \in K_m$ and $L' \subset L$, with $L \cap E \neq \emptyset$ and $L' \cap E = \emptyset$, then $L' \in K_m$. In this case, there exists $e \in E \cap (L\backslash L')$ so $m - n_{L'} = m - n_L + n_{L\backslash L'} = m - n_L + e + s$, for $s = n_{L\backslash(L'\cup\{e\})} \in S$, so $m - n_{L'} \notin \mathcal{B}$, so again $L' \in K_m$.

The above definition for K_m is given in [14], and describes all faces of K_m. From this definition, it is clear that K_m can never equal $\{\emptyset\}$. Indeed, no face of Δ_m meets E if and only if $m \in \mathcal{B}$ if and only if $K_m = \{\}$. The last equivalence also implies that $m \in \mathcal{B}$ if and only if $\tilde{C}.(K_m)$ is the zero complex. We now give an alternative description in terms of the facets of Δ_m.

Proposition 5.4.1. *With the notation above, suppose there exists facets of Δ_m which intersect E nontrivially. Then K_m is the simplicial subcomplex of Δ_m generated by these facets. Otherwise $K_m = \{\}$. In particular, if $\{F_1, \ldots, F_l\}$ are the facets of Δ_m which intersect E nontrivially then these are the facets of K_m. If no such facets exist then $K_m = \{\}$.*

Proof. Clearly all of the facets of Δ_m which intersect E nontrivially are facets of K_m. It remains to show that there are no more. Suppose L is a facet of K_m such that $L \cap E = \emptyset$. Then by definition of K_m, $m - n_L \in S\backslash\mathcal{B}$. This implies, that $m - n_L - e \in S$ for some $e \in E$. Thus $L \cup \{e\}$ is a face of Δ_m which intersects E nontrivially. Hence $L \cup \{e\}$ is a face of K_m. This contradicts the maximality of L. ∎

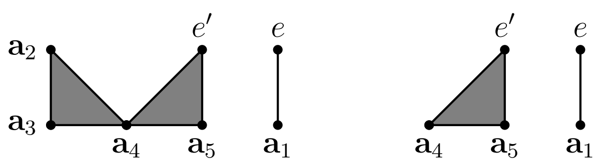
The description of K_m given in Proposition 5.4.1 gives a clear description of K_m in terms of Δ_m. Moreover, given the facets of Δ_m it is trivial to compute K_m.

We also have the following consequence.

Corollary 5.4.2. *Let $m \in S$ and suppose m cannot be written as $m = q + n_I$ for some $q \in \mathcal{B}$, $I \subseteq A$. Then $\Delta_m = K_m$.*

Proof. If $\Delta_m \neq K_m$, then some facet I of Δ_m does not meet E. Hence $m - n_I \in \mathcal{B}$ for some $I \subseteq A$ which is a contradiction. □

Example 5.4.3. If Δ_m is defined by facets $\{\{e, \mathbf{a}_1\}, \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}, \{e', \mathbf{a}_5, \mathbf{a}_4\}\}$ then we have that K_m is defined by facets $\{\{e, \mathbf{a}_1\}, \{e', \mathbf{a}_5, \mathbf{a}_4\}\}$. The geometric realization of Δ_m is pictured to the left below. That of K_m is pictured to the right.



Consider now the set, [14, p. 147] (in this definition we are assuming $|J| \geq 2$):

$$\bar{K}_m = K_m \cup \{I \cup J \mid I \subseteq A, J \subseteq E,\ m - n_I - n_J \notin S \text{ and } m - n_I - e \in S, \forall e \in J\}.$$

We claim that $\bar{K}_m$ is a simplicial complex containing K_m. It is clear that as sets $K_m \subseteq \bar{K}_m$. It remains to show that $\bar{K}_m$ is closed under taking subsets.

Clearly, if $L \in \bar{K}_m$ and $L \in K_m$ then any subset of L is in $\bar{K}_m$ as K_m is a simplicial complex. Now suppose that $L = I \cup J \in \bar{K}_m \backslash K_m$ (this implies that $|J| \geq 2$). Let $L' \subset L$. Then we may partition L' as $L' = I' \cup J'$ with $I' \subset I \subset A$ and $J' \subset J \subset E$. Since $L \in \bar{K}_m \backslash K_m$, we have that $m - n_I - e \in S$ for all $e \in J$ but $m - n_I - n_J \notin S$. This implies that $m - n_{I'} - e \in S$ for any $e \in J'$. Hence if $m - n_{L'} \notin S$ (i.e.,

$m - n_{I'} - n_{J'} \notin S$), then we must have $|J'| \geq 2$ so that $L' \in \bar{K}_m \backslash K_m$. On the other hand, if $m - n_{L'} \in S$ then $L' \in \Delta_m$. If $J' \neq \emptyset$, then $L' \cap E \neq \emptyset$. If $J' = \emptyset$, then since $m - n_{I'} - e \in S$ for all $e \in J$, we have that $m - n_{I'} \in S \backslash \mathcal{B}$. In either case $L' \in K_m$ so that $L' \in \bar{K}_m$. Thus $\bar{K}_m$ is a simplicial complex as claimed.

Moreover, if we are in two dimensions we have that $J = E$, so that the definition of $\bar{K}_m$ simplifies to:

$$\bar{K}_m = K_m \cup \{I \cup E, \mid I \subseteq A, m - n_I - n_E \notin S \text{ and } m - n_I - e \in S, \forall e \in E\}.$$

Translating the above definition into words, we also have the following description of $\bar{K}_m$. (In the two dimensional case.)

Proposition 5.4.4. *The facets of $\bar{K}_m$ are the maximal sets of the following set:*

$$\{\textit{Facets of } K_m\} \cup$$

$$\{I \cup E \mid I \subseteq A, \textit{ such that } I \cup \{e\} \textit{ and } I \cup \{e'\} \textit{ are faces of } K_m$$

$$\textit{but } I \cup \{e, e'\} \textit{ is not a face of } K_m.\}.$$

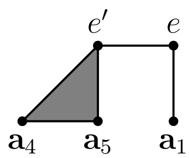
Proof. All the facets of K_m are faces of $\bar{K}_m$, although they may cease to be facets. On the other hand, all new faces of $\bar{K}_m$ (i.e., the faces of $\bar{K}_m$ which are not faces of K_m) are contained in the right hand side of the union. □

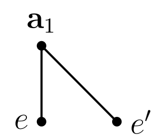
The definition of $\bar{K}_m$, as is, may seem strange. We now present several examples illustrating, what it may look like. The examples should help convince the reader that $\bar{K}_m$ is acyclic, which is shown to be the case in [14, Proposition 2.2, p. 148], and for emphasis we state:

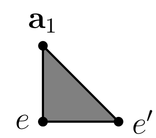
Proposition 5.4.5 (Proposition 2.2, p. 148 [14])**.** *The simplicial complex $\bar{K}_m$ is acyclic.*

Note also that if $m \in \mathcal{B}$ then $\bar{K}_m = \{\}$. Moreover note that both K_m and $\bar{K}_m$ have the same zero-dimensional faces.

Examples 5.4.6. If K_m is defined as in Example of 5.4.3 then $\bar{K}_m$ is defined by facets $\{\{e, e'\}, \{e, \mathbf{a}_1\}, \{e', \mathbf{a}_5, \mathbf{a}_4\}\}$ and pictured on the left below. If K_m is defined by the facets $\{\{\mathbf{a}_1, e\}, \{\mathbf{a}_1, e'\}\}$ then $\bar{K}_m$ is defined by a single facet $\{\{e, e', \mathbf{a}_1\}\}$. In this case K_m is pictured in the middle below and $\bar{K}_m$ is pictured on the right.

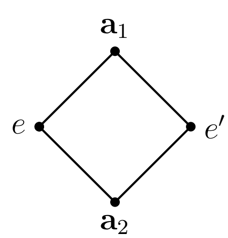


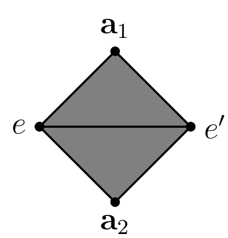




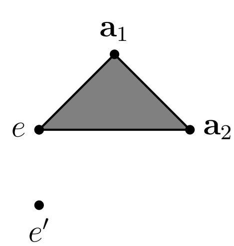
Examples 5.4.7.

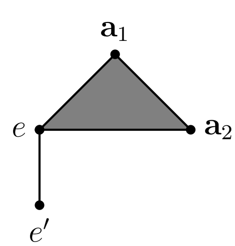
- If K_m is defined by facets $\{\{e, \mathbf{a}_1\}, \{\mathbf{a}_1, e'\}, \{\mathbf{a}_2, e'\}, \{e, \mathbf{a}_2\}\}$ then $\bar{K}_m$ is defined by facets $\{\{e, e', \mathbf{a}_1\}, \{e, e', \mathbf{a}_2\}\}$.





- If K_m is defined by facets $\{\{e, \mathbf{a}_1, \mathbf{a}_2\}, \{e'\}\}$ then $\bar{K}_m$ is defined by facets $\{\{e, \mathbf{a}_1, \mathbf{a}_2\}, \{e, e'\}\}$.





5.5 The relative homology $\tilde{H}_t(\bar{K}_m, K_m)$

In Section 5.4 we presented the definitions of K_m and $\bar{K}_m$ in a more general context. As such, we now remind the reader that we are still assuming that $S \in \mathscr{C}'$ so that $S \subseteq \mathbb{N}^2$ and has minimal generating set Λ constructed from $\mathscr{S} = \{m_1, \ldots, m_n\}$ as in Section 5.1.

We would like to compute $\tilde{H}_t(\bar{K}_m, K_m)$. In order to do so we must make some more definitions. Set $M_m^{(-1)} := K_m$ and, for each $0 \leq i \leq r := |A| = n - 1$, define $M_m^{(i)}$ to be the simplicial subcomplex of $\bar{K}_m$:

$$M_m^{(i)} = K_m \cup \{L = I \cup J \in \bar{K}_m \mid I \subseteq A, J \subseteq E \text{ and } |I| \leq i\}.$$

It is clear from the definitions that for all i, $-1 \leq i \leq r$, $M_m^{(i)}$ is a simplicial complex (any subset of K_m is an element of $M_m^{(i)}$ and any subset of $L = I \cup J \subseteq M_m^{(i)} \backslash K_m$ is a face of $\bar{K}_m$ and contains less than or equal to $|I|$ elements of A so will also be a face of $M_m^{(i)}$). In other words, $M_m^{(i)}$ is the simplicial complex consisting of the faces of K_m together with the faces of $\bar{K}_m$ containing less than or equal to i elements of A. We now have the following filtration of simplicial complexes:

$$K_m = M_m^{(-1)} \subseteq M_m^{(0)} \subseteq M_m^{(1)} \subseteq \cdots \subseteq M_m^{(r)} = \bar{K}_m.$$

The above filtration implies that for any triple $(i, j, k), 1 \leq i < j < k \leq r$ the short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}.(M_m^{(j)}, M_m^{(i)}) \longrightarrow \tilde{C}.(M_m^{(k)}, M_m^{(i)}) \longrightarrow \tilde{C}.(M_m^{(k)}, M_m^{(j)}) \longrightarrow 0$$

give rise to the long exact sequence in homology:

$$\cdots \longrightarrow \tilde{H}_t(M_m^{(j)}, M_m^{(i)}) \longrightarrow \tilde{H}_t(M_m^{(k)}, M_m^{(i)}) \longrightarrow \tilde{H}_t(M_m^{(k)}, M_m^{(j)}) \longrightarrow \cdots,$$

$t \geq -1$.

We would like to determine when $\tilde{H}_t(\bar{K}_m, K_m) = \tilde{H}_t(M_m^{(r)}, M_m^{(-1)}) = 0$ for some $t \geq -1$. In order to do this, by recursion, it is enough to use triples $(i, j, k) = (-1, 0, 1), (-1, 1, 2), \ldots, (-1, r-1, r)$. We show this momentarily.

Since we are taking relative homology we always have $\tilde{H}_{-1}(M_m^{(i)}, M_m^{(i-1)}) = 0$ and $\tilde{H}_{-1}(\bar{K}_m, K_m) = 0$. We now give a sufficient condition for $\tilde{H}_t(\bar{K}_m, K_m) = 0$, $t \geq 0$.

Lemma 5.5.1. *Fix $t \geq 0$. If for all $i = 0, 1, \ldots, r$ we have $\tilde{H}_t(M_m^{(i)}, M_m^{(i-1)}) = 0$ then $\tilde{H}_t(\bar{K}_m, K_m) = 0$.*

Proof. The exact sequence

$$0 \longrightarrow \tilde{C}.(M_m^{(r-1)}, M_m^{(-1)}) \longrightarrow \tilde{C}.(M_m^{(r)}, M_m^{(-1)}) \longrightarrow \tilde{C}.(M_m^{(r)}, M_m^{(r-1)}) \longrightarrow 0$$

of chain complexes induces the long exact sequence

$$\cdots \longrightarrow \tilde{H}_t(M_m^{(r-1)}, M_m^{(-1)}) \longrightarrow \tilde{H}_t(M_m^{(r)}, M_m^{(-1)}) \longrightarrow \tilde{H}_t(M_m^{(r)}, M_m^{(r-1)}) \longrightarrow \cdots$$

on homology. By assumption, $\tilde{H}_t(M_m^{(r)}, M_m^{(r-1)}) = 0$. It remains to show that the assumptions also imply that $\tilde{H}_t(M_m^{(r-1)}, M_m^{(-1)}) = 0$ whence exactness will imply that $\tilde{H}_t(M_m^{(r)}, M_m^{(-1)}) = 0$ thus, completing the proof.

To compute $\tilde{H}_t(M_m^{(r-1)}, M_m^{(-1)})$ we may consider a triple of the form $(-1, r-2, r-1)$. Then, as above, we obtain a long exact sequence of homology such that the right hand term, $\tilde{H}_t(M_m^{(r-1)}, M_m^{(r-2)}) = 0$ by assumption. Continuing this process, by recursion, we obtain an exact sequence, corresponding to the triple $(-1, 0, 1)$:

$$\cdots \longrightarrow \tilde{H}_t(M_m^{(0)}, M_m^{(-1)}) \longrightarrow \tilde{H}_t(M_m^{(1)}, M_m^{(-1)}) \longrightarrow \tilde{H}_t(M_m^{(1)}, M_m^{(0)}) \longrightarrow \cdots$$

whereby, the assumptions imply that, $\tilde{H}_t(M_m^{(1)}, M_m^{(-1)}) = 0$. Backwards substitution now completes the proof. $\square$

By definition the vertices of $M_m^{(i)}$ and $M_m^{(i-1)}$ are the same. This implies that $\tilde{C}_0(M_m^{(i)}, M_m^{(i-1)}) = 0$ and $\tilde{H}_0(M_m^{(i)}, M_m^{(i-1)}) = 0$. We now give an explicit description of a $\mathbb{K}$-basis for $\tilde{C}_t(M_m^{(i)}, M_m^{(i-1)})$. By definition of $M_m^{(i)}$ and $M_m^{(i-1)}$, and after canonically identifying elements of the quotient, $\tilde{C}_t(M_m^{(i)}, M_m^{(i-1)}) := \tilde{C}_t(M_m^{(i)})/\tilde{C}_t(M_m^{(i-1)})$, it is clear from the definitions that a $\mathbb{K}$-basis for $\tilde{C}_t(M_m^{(i)}, M_m^{(i-1)})$ is given by the set:

$$\{\mathbf{e}_L \mid L = I \cup E \in \bar{K}_m \text{ such that } I \subseteq A,\ |L| = t+1,\ |I| = i, \text{ and } L \text{ is not a face of } K_m\}.$$

This is just saying that a $\mathbb{K}$-basis for $\tilde{C}_t(M_m^{(i)}, M_m^{(i-1)})$ can identified with the canonical image in the quotient of the basis elements $\mathbf{e}_L$ of $\tilde{C}_t(M_m^{(i)})$ such that $L = I \cup E$ contains i elements of A and L is not a face of K_m. Moreover, the above also implies that if $\mathbf{e}_L$ is a basis element of $\tilde{C}_t(M_m^{(i)}, M_m^{(i-1)})$ then $|L| = t + 1 = i + 2$. In particular, $C_t(M_m^{(i)}, M_m^{(i-1)}) = 0$ for $t \neq i + 1$.

Considering now the relative chain complex $\tilde{C}.(M_m^{(i)}, M_m^{(i-1)})$ we note that, since $|E| = 2$, $\tilde{C}.(M_m^{(i)}, M_m^{(i-1)})$ reduces to a short exact sequence:

$$0 \longrightarrow \tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \longrightarrow 0\ ,$$

so that $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) = \tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)})$ and $\tilde{H}_j(M_m^{(i)}, M_m^{(i-1)}) = 0$ for all $j \geq 0, j \neq i + 1$.

This observation will be important in what follows. Hence, for emphasis, we state the following lemma.

Lemma 5.5.2. *With the notation as above, for all $j \geq 0, j \neq i + 1$ we have $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) = \tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)})$, and $\tilde{H}_j(M_m^{(i)}, M_m^{(i-1)}) = 0$.*

To help the reader understand the above definitions and notation, we do an example.

Example 5.5.3. Let $\mathscr{S} = \{7, 10, 13, 15, 24, 25\}$ so that S is minimally generated by

$$\Lambda = \{\{25, 0\}, \{18, 7\}, \{15, 10\}, \{12, 13\}, \{10, 15\}, \{1, 24\}, \{0, 25\}\}.$$

Let $m = \{55, 95\}$. We then compute that Δ_m is defined by the facets:

$$\{\{1, 3, 7\}, \{4, 5, 6\}, \{1, 2, 4, 7\}, \{1, 3, 4, 6\}, \{2, 3, 5, 6\},$$

$$\{2, 4, 6, 7\}, \{3, 4, 6, 7\}, \{1, 2, 5, 6, 7\}, \{2, 3, 4, 5, 7\}\}.$$

Using Proposition 5.4.1 it is easy to see that $K_m = M_m^{(-1)}$ is defined by the facets:

$$\{\{1, 3, 7\}, \{1, 2, 4, 7\}, \{1, 3, 4, 6\},$$

$$\{2, 4, 6, 7\}, \{3, 4, 6, 7\}, \{1, 2, 5, 6, 7\}, \{2, 3, 4, 5, 7\}\}.$$

Since $\{1, 7\}$ is a face of K_m, we have that $M_m^{(0)} = K_m$. Since $\{1, a, 7\}$ is a face of K_m, $2 \leq a \leq 6$, we also have that $M_m^{(1)} = K_m$.

On the other hand $M_m^{(2)}$ is given by:

$$M_m^{(2)} := K_m \cup \{\{1, 3, 4, 7\}, \{1, 3, 6, 7\}, \{1, 4, 6, 7\}\},$$

since for example, $\{3, 4, 7\}$ and $\{1, 3, 4\}$ are faces of K_m but $\{1, 3, 4, 7\}$ is not a face of K_m. Similarly, one checks that $\{1, 3, 6, 7\}$ and $\{1, 4, 6, 7\}$ are faces of $M_m^{(2)}$ not contained in K_m. Considering all four element subsets of $\{1, \ldots, 7\}$ containing both 1 and 7, we see that these are all of the new faces.

In a similar fashion, one computes that $M_m^{(3)}$ is given by

$$M_m^{(3)} := M_m^{(2)} \cup \{\{1, 3, 4, 6, 7\}\},$$

and that $M_m^{(3)} = \bar{K}_m$.

Considering $\tilde{C}.(M_m^{(2)}, M_m^{(1)})$, we have:

$$0 \longrightarrow \tilde{C}_3(M_m^{(2)}, M_m^{(1)}) \longrightarrow 0 \,,$$

where a $\mathbb{K}$-basis of $\tilde{C}_3(M_m^{(2)}, M_m^{(1)})$ is given by $\{\mathbf{e}_{\{1,3,4,7\}}, \mathbf{e}_{\{1,3,6,7\}}, \mathbf{e}_{\{1,4,6,7\}}\}$.

Similarly considering $\tilde{C}.(M_m^{(3)}, M_m^{(2)})$, we have:

$$0 \longrightarrow \tilde{C}_4(M_m^{(3)}, M_m^{(2)}) \longrightarrow 0 \,,$$

with a $\mathbb{K}$-basis of $\tilde{C}_4(M_m^{(3)}, M_m^{(2)})$ given by $\{\mathbf{e}_{\{1,3,4,6,7\}}\}$.

To see that $\tilde{C}_3(M_m^{(3)}, M_m^{(2)}) = 0$, for example, we note that the faces:

$$\{1,3,6,7\}, \{1,4,6,7\}, \{1,3,4,7\}$$

of $M_m^{(3)}$ are elements of $M_m^{(2)} \backslash K_m$ whereas $\{3,4,6,7\}$ and $\{1,3,4,6\}$ are faces of K_m.

5.6 A vanishing theorem

Our goal now is to understand [14, Proposition 3.2, p. 153]. In light of Lemma 5.5.1, we have the following criterion: If for all $i = 0, 1, \ldots, r = |A| = n-1$ we have $\tilde{H}_t(M_m^{(i)}, M_m^{(i-1)}) = 0$ then $\tilde{H}_t(\bar{K}_m, K_m) = 0$. To use this criterion, we establish a correspondence between $\tilde{H}_t(M_m^{(i)}, M_m^{(i-1)})$, and the homology of certain simplicial complexes T_m ([14, p. 153]) which we now define.

For every $m \in S$ define the simplicial complex on the vertex set E as follows:

$$T_m := \{J \subseteq E \mid m - n_J \in S\}.$$

Moreover, set

$$E_m := \{e \in E \mid m - e \in S\},$$

and define Σ'_m to be the simplicial complex consisting of all subsets of E_m. Then T_m is a simplicial subcomplex of Σ'_m. If $m \in \mathcal{B}$ then $E_m = \emptyset$, $\Sigma'_m = \{\emptyset\}$ and $T_m = \{\emptyset\}$. If $m \notin \mathcal{B}$ then, since $E = \{e, e'\}$, we have that Σ'_m is defined by one of the three sets of facets: $\{\{e\}\}$, $\{\{e'\}\}$, or $\{\{e, e'\}\}$ and that there are four choices for T_m: $\{\{e\}\}$, $\{\{e'\}\}$, $\{\{e\}, \{e'\}\}$, or $\{\{e, e'\}\}$.

We now relate $\tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)})$ to the simplicial complexes T_m and Σ'_m just defined. Since the vertices of Σ'_{m-n_I} and T_{m-n_I} are the same by definition we have that $\tilde{C}_0(\Sigma'_{m-n_I}, T_{m-n_I}) = 0$, which implies that for all $I \subseteq A$ such that $m - n_I \in S$ we have $\tilde{H}_0(\Sigma'_{m-n_I}, T_{m-n_I}) = 0$. Hence, $\tilde{C}.(\Sigma'_{m-n_I}, T_{m-n_I})$ reduces to:

$$0 \longrightarrow \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}) \longrightarrow 0\,.$$

This implies that $\tilde{H}_1(\Sigma'_{m-n_I}, T_{m-n_I}) = \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I})$ and that $\tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}) \neq 0$ if and only if $m - n_I - e \in S$, $m - n_I - e' \in S$ and $m - n_I - n_E \notin S$. (This is the only case in which we will have $\Sigma'_{m-n_I} \neq T_{m-n_I}$.)

Consider now the $\mathbb{K}$-vector space

$$\bigoplus_{I \subseteq A, |I| = i, m - n_I \in S} \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}).$$

We can denote a $\mathbb{K}$-basis for the sum as the set $\{\mathbf{e}_I \mid I \subseteq A$ and $|I| = i$ and $\tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}) \neq 0\}$.

I now claim that:

$$\tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \cong \bigoplus_{I \subseteq A, |I| = i, m - a_I \in S} \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}),$$

via the isomorphism $\mathbf{e}_L \mapsto \mathbf{e}_I$ (recall that $L = I \cup E$).

Indeed, by the previous discussion, we have that a $\mathbb{K}$-basis for $\tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)})$ is given by the set:

$\{\mathbf{e}_L \mid L = I \cup E \in \bar{K}_m$ such that $I \subseteq A$, $|L| = i + 2$, $|I| = i$, and L is not a face of $K_m\}$.

On the other hand, the definitions of $\bar{K}_m$ and $M_m^{(i)}$ imply that for a fixed $L = I \cup E \in \bar{K}_m \backslash K_m$, $I \subseteq A, |I| = i$, (so that L is not a face of $M_m^{(i-1)}$) we have, by definition of $\bar{K}_m$, that $m - n_I - e \in S$ and $m - n_I - e' \in S$ but $m - n_I - n_E \notin S$. Hence the correspondence $\mathbf{e}_L \mapsto \mathbf{e}_I$ is bijective.

We have the following statement.

Proposition 5.6.1. *With the notation and assumptions above, for a fixed i, $0 \leq i \leq r = |A| = n - 1$, we have the following isomorphisms on homology for all $j \geq 0$:*

1. $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \cong \bigoplus_{I \subseteq A, |I| = i, m - n_I \in S} \tilde{H}_0(T_{m-n_I})$.

2. $\tilde{H}_j(M_m^{(i)}, M_m^{(i-1)}) = 0$, $j \neq i + 1$.

Proof. The second assertion is contained in Lemma 5.5.2. To prove the first, by Lemma 5.5.2, $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) = \tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)})$. Combing this with the previous observation that $\tilde{H}_1(\Sigma'_{m-n_I}, T_{m-n_I}) = \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I})$, along with the above discussion, we have that

$$\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \cong \bigoplus_{I \subseteq A, |I| = i, m - n_I \in S} \tilde{H}_1(\Sigma'_{m-n_I}, T_{m-n_I}).$$

We now observe that the $\mathbb{K}$-vector space $\tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}) \neq 0$ exactly when $m - n_I - \{e\} \in S$, $m - n_I - \{e'\} \in S$, $m - n_I - n_E \notin S$, and is 1-dimensional. This is exactly the case when $T_{m-n_I} = \{\{e\}, \{e'\}\}$, i.e., the only case when $\tilde{H}_0(T_{m-n_I})$ is 1-dimensional. $\square$

Remark 5.6.2. The above has been extracted from page 152 to the bottom of page 153 of [14]. In particular, Proposition 5.6.1, is more or less [14, Proposition 3.2, p. 153], although we hope the above discussion has helped to explain what is happening.

To help the reader understand the above notation and proof, we do an example.

Example 5.6.3. This example is a continuation of Example 5.5.3. Recall that S is minimally generated by

$$\Lambda = \{\{25,0\},\{18,7\},\{15,10\},\{12,13\},\{10,15\},\{1,24\},\{0,25\}\},$$

and that $m = \{55,95\}$. We illustrate the assertion that

$$\tilde{C}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \cong \bigoplus_{I\subseteq A, |I|=i, m-n_I\in S} \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I}).$$

It is easy to compute T_{m-n_I} for all I in the sum using K_m. For example, let $i = 2$. We summarize Σ'_{m-n_I} and T_{m-n_I} in Table 5.1.

We have that T_{m-n_I} is disconnected for $I \in \{\{3,4\},\{3,6\},\{4,6\}\}$, so that a $\mathbb{K}$-basis for

$$\bigoplus_{I\subseteq A, |I|=2, m-n_I\in S} \tilde{C}_1(\Sigma'_{m-n_I}, T_{m-n_I})$$

is given by $\{\mathbf{e}_{\{3,4\}}, \mathbf{e}_{\{3,6\}}, \mathbf{e}_{\{4,6\}}\}$. Comparing this with the $\mathbb{K}$-basis for $\tilde{C}_3(M_m^{(2)}, M_m^{(1)})$, described in Example 5.5.3, we see that the correspondence $\mathbf{e}_L \mapsto \mathbf{e}_I$ is an isomorphism.

5.7 A set containing all nontrivial Betti numbers

In this section, for a fixed $t \geq 0$, we give a finite subset of S containing all $m \in S$ such that $\tilde{H}_t(\Delta_m) \neq 0$. We first recall what exactly we are doing. We would like to give a sufficient condition for $\tilde{H}_t(\Delta_m) = 0$ for some $t \geq 0$. Recall that we have the following long exact sequence of homology:

$$\cdots \longrightarrow \tilde{H}_t(K_m) \longrightarrow \tilde{H}_t(\Delta_m) \longrightarrow \tilde{H}_t(\Delta_m, K_m) \longrightarrow \cdots,$$

I	T_{m-n_I}	Σ'_{m-n_I}
{2,3}	{{7}}	{{7}}
{2,4}	{{1,7}}	{{1,7}}
{2,5}	{{1,7}}	{{1,7}}
{2,6}	{{1,7}}	{{1,7}}
{3,4}	{{1},{7}}	{{1,7}}
{3,5}	{{7}}	{{7}}
{3,6}	{{1},{7}}	{{1,7}}
{4,5}	{{7}}	{{1,7}}
{4,6}	{{1},{7}}	{{1,7}}
{5,6}	{{1,7}}	{{1,7}}

Table 5.1: The simplicial complexes T_{m-n_I} and Σ'_{m-n_I} of Example 5.6.3.

and the relation $\tilde{H}_t(K_m) \cong \tilde{H}_{t+1}(\bar{K}_m, K_m)$. This implies that if $\tilde{H}_{t+1}(\bar{K}_m, K_m) = 0$ and $\tilde{H}_t(\Delta_m, K_m) = 0$ then $\tilde{H}_t(\Delta_m) = 0$.

We now give a sufficient condition for $\tilde{H}_t(\Delta_m, K_m) = 0$ for all $t \geq 0$.

Proposition 5.7.1. *Let $m \in S$ and suppose $m - n_I \notin \mathcal{B}$ for all $I \subseteq A$. Then $\tilde{H}_t(\Delta_m, K_m) = 0$ for all $t \geq 0$.*

Proof. If $m - n_I \notin \mathcal{B}$ for all $I \subseteq A$, then all of the facets of Δ_m meet E so that $\Delta_m = K_m$ whence $\tilde{C}.(\Delta_m, K_m)$ equals the zero complex. Thus, $\tilde{H}_t(\Delta_m, K_m) = 0$ for all $t \geq 0$. □

In fact, we can be more specific. If $\tilde{C}_t(\Delta_m) = \tilde{C}_t(K_m)$ then $\tilde{C}_t(\Delta_m, K_m) = 0$ and thus $\tilde{H}_t(\Delta_m, K_m) = 0$. We take care of this in the following statement.

Proposition 5.7.2. *Let $m \in S$ and suppose that $m - n_I \notin \mathcal{B}$ for all $I \subseteq A$, such that $|I| = t + 1$. Then $\tilde{H}_t(\Delta_m, K_m) = 0$.*

Proof. If $m - n_I \notin \mathcal{B}$ for all $I \subseteq A$ such that $|I| = t + 1$, then all of the t-dimensional faces of Δ_m are contained in a facet meeting E so that $\tilde{C}_t(\Delta_m) = \tilde{C}_t(K_m)$. Thus $\tilde{C}_t(\Delta_m, K_m) = 0$, so that $\tilde{H}_t(\Delta_m, K_m) = 0$. □

Let us now determine when $\tilde{H}_t(\bar{K}_m, K_m) = 0$ for some $t \geq 1$.

Proposition 5.7.3. *With the notation above, fix $t \geq 0$. If for all $I \subseteq A$, such that $|I| = t$ and such that $m - n_I \in S$, we have that $\tilde{H}_0(T_{m-n_I}) = 0$ then $\tilde{H}_{t+1}(\bar{K}_m, K_m) = 0$.*

Proof. Lemma 5.5.1 implies that if, for all $i = 0, 1, \ldots, r = |A| = n - 1$, we have $\tilde{H}_t(M_m^{(i)}, M_m^{(i-1)}) = 0$ then $\tilde{H}_t(\bar{K}_m, K_m) = 0$. Moreover, Proposition 5.6.1 says that for all $i \geq 0$,

$$\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) \cong \bigoplus_{I \subseteq A, |I| = i, m - n_I \in S} \tilde{H}_0(T_{m-n_I}),$$

and that $\tilde{H}_t(M_m^{(i)}, M_m^{(i-1)}) = 0$ for all $t \neq i + 1$. It thus follows (by Lemma 5.5.1) that if, for a fixed $i \geq 0$, we have $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) = 0$ then $\tilde{H}_{i+1}(\bar{K}_m, K_m) = 0$. The above isomorphism implies that $\tilde{H}_{i+1}(M_m^{(i)}, M_m^{(i-1)}) = 0$ if and only if for all $I \subseteq A$, such that $|I| = t$ and, such that, $m - n_I \in S$, we have that $\tilde{H}_0(T_{m-n_I}) = 0$. This final observation completes the proof. $\square$

Recall that, just before Proposition 5.7.1, we derived the following criterion: if $\tilde{H}_{t+1}(\bar{K}_m, K_m) = 0$ and $\tilde{H}_t(\Delta_m, K_m) = 0$ then $\tilde{H}_t(\Delta_m) = 0$. The contrapositive reads: if $\tilde{H}_t(\Delta_m) \neq 0$ then $\tilde{H}_{t+1}(\bar{K}_m, K_m) \neq 0$ or $\tilde{H}_t(\Delta_m, K_m) \neq 0$. It thus follows that the multidegrees $m \in S$ such that $\tilde{H}_t(\Delta_m) \neq 0$, $t \geq 0$ are contained in the set:

$$\mathscr{B}_t := \{m \in S \mid \tilde{H}_{t+1}(\bar{K}_m, K_m) \neq 0 \text{ or } \tilde{H}_t(\Delta_m, K_m) \neq 0\}.$$

We hope that $\mathscr{B}_t$ is finite. At present we have developed enough theory to describe a finite set, which we label C_t, containing $\mathscr{B}_t$.

Theorem 5.7.4. *The multidegrees of all t-syzygies of $\mathfrak{p}$ are contained in the finite set*

$$C_t := \{\mathcal{B} + \textit{sums of } t + 1 \textit{ distinct elements of } A\}$$

$$\cup\{(d,d) + \ \tilde{S}\backslash S\ +\ \textit{sums of } t \textit{ distinct elements of } A\}\ .$$

(Since $|A| = n-1$*, we have that* $C_{n-1} = \{n_\Lambda + \tilde{S}\backslash S\}$*, and that* $C_t = \{\}, t \geq n$*.)*

Proof. By the previous discussion all multidegrees of t-syzygies are contained in $\mathscr{B}_t$. Thus, we may show that $\mathscr{B}_t \subseteq C_t$ and that C_t is finite. Proposition 5.7.2 implies that if $m \in S$ and $\tilde{H}_t(\Delta_m, K_m) \neq 0$ then $m - n_I \in \mathcal{B}$ for some $I \subseteq A$ with $|I| = t+1$. This implies that for some $I \subseteq A$, $|I| = t+1$ and for some $q \in \mathcal{B}$ we have $m = q + n_I$. Thus, the elements of $m \in \mathscr{B}_t$ with $\tilde{H}_t(\Delta_m, K_m) \neq 0$ are contained in the set $\{\mathcal{B} + \text{sums of } t+1 \text{ distinct elements of } A\}$.

Before proceeding further, we note that the elements $m' \in S$ such that $\tilde{H}_0(T_{m'}) \neq 0$ are of the form $m' = s + n_E$ for some $s \in \tilde{S}\backslash S$. Indeed, by definition of $\tilde{S}$ it is clear that if, for some $s \in \tilde{S}\backslash S$, we have $m' = s + (d,d)$ then $\tilde{H}_0(T_{m'}) \neq 0$. Conversely, let $m' \in S$ and suppose that $\tilde{H}_0(T_{m'}) \neq 0$. Then $m' - e \in S$ and $m' - e' \in S$ but $m' - n_E = s \in G(S)\backslash S$. Since $s + e = m' - e' \in S$ and $s + e' = m' - e \in S$, we have $s \in \tilde{S}$. It follows that if $m \in S$ and $m - n_I \in S$ with $\tilde{H}_0(T_{m-n_I}) \neq 0$, for some $I \subseteq A$ such that $|I| = t$, then $m = n_I + s + (d,d)$.

Now let $m \in S$ and suppose that $\tilde{H}_{t+1}(\bar{K}_m, K_m) \neq 0$. Then, by Proposition 5.7.3, there exists $I \subseteq A$ such that $|I| = t$ and such that $m - n_I \in S$ with $\tilde{H}_0(T_{m-n_I}) \neq 0$. We just showed that this implies that $m = n_I + s + (d,d)$ for some $s \in \tilde{S}\backslash S$. Thus, the elements of $m \in \mathscr{B}_t$ with $\tilde{H}_{t+1}(\bar{K}_m, K_m) \neq 0$ are contained in the set $\{(d,d) + \tilde{S}\backslash S + \text{sums of } t \text{ distinct elements of } A\}$. We have shown that $\mathscr{B}_t \subseteq C_t$. Since both $\mathcal{B}$ and $\tilde{S}\backslash S$ are finite sets so is C_t. $\square$

We would like to relate Theorem 5.7.4 with the terminology of Chapter 4 which we now recall. Recall, that the second coordinates of elements of $\tilde{S}\backslash S$ are immediately unstable elements of $\mathrm{gr}(\mathscr{S})$ (i.e., if x is the second coordinate of some element of

$\tilde{S}\backslash S$ then t^x is killed by t^d in $\mathrm{gr}(\mathscr{S})$) and, similarly, the first coordinates of $\tilde{S}\backslash S$ are the immediately unstable elements of $\mathrm{gr}(\hat{\mathscr{S}})$. Recall that if $x = (x_1, x_2) \in \tilde{S}\backslash S$ then $\deg(x) = \mathrm{ord}(x_2) - 1 = \mathrm{ord}(x_1) - 1$, and that the maximum order of immediately unstable elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ coincide (Lemma 4.2.13).

We also recall that $x = (x_1, x_2) \in (\mathcal{B})_i$, if and only if the canonical image of t^{x_2} in the quotient $\mathrm{gr}(\mathscr{S})_i/((t^d)\,\mathrm{gr}(\mathscr{S})_{i-1})$ is non-zero (Proposition 4.2.4). Moreover, the canonical lift of the monomials of $\mathrm{gr}(\mathscr{S})/(t^d)\,\mathrm{gr}(\mathscr{S})$ constitutes a finite (minimal) generating set of $\mathrm{gr}(\mathscr{S})$ as a $\mathbb{K}[t^d]$-module, as does the lift of $\mathrm{gr}(\hat{\mathscr{S}})/(s^d)\,\mathrm{gr}(\hat{\mathscr{S}})$ for $\mathrm{gr}(\hat{\mathscr{S}})$ as a $\mathbb{K}[s^d]$-module. Thus, we may refer to the second coordinate of elements of $\mathcal{B}$ as the "basis" of $\mathrm{gr}(\mathscr{S})$ and the first coordinate of elements of $\mathcal{B}$ as the "basis" of $\mathrm{gr}(\hat{\mathscr{S}})$ (Definition 4.2.6).

The following observation is immediate from the definitions and useful in what follows.

Lemma 5.7.5. *The maximum $\mathbb{N}$-degree of an element of C_0 is equal to:*

$$\max\{\{\max\{\mathbb{N}\text{-}\textit{degree of a basis element of } S\} + 1\} \cup$$

$$\{\max\{\mathbb{N}\text{-}\textit{degree of an element of } \tilde{S}\backslash S\} + 2\}\}.$$

Equivalently, the maximum degree of an element of C_0 is equal to:

$$\max\{\max\{\mathrm{ord}(x) \mid t^x \textit{ is a basis element of } \mathrm{gr}(\mathscr{S})\} + 1\} \cup \{\max\{\mathrm{ord}(x) \mid x$$

$$\textit{is an immediately unstable element of } \mathrm{gr}(\mathscr{S})\} + 1\}\}.$$

Example 5.7.6. Let $\mathscr{S} = \{2, 12, 15\}$, so that $\Lambda = \{\{15, 0\}, \{13, 2\}, \{3, 12\}, \{0, 15\}\}$. Using *Mathematica*, the sets C_i were found to be:

```
C_0:={{3, 12}, {6, 24}, {9, 36}, {12, 48}, {13, 2}, {15, 60}, {16,
  14}, {19, 26}, {22, 38}, {25, 50}, {26, 4}, {28, 62}, {29, 16}, {32,
   28}, {35, 40}, {38, 52}, {39, 6}, {39, 36}, {41, 64}, {42,
  18}, {42, 48}, {45, 30}, {48, 42}, {51, 54}, {52, 8}, {52, 38}, {55,
   20}, {55, 50}, {58, 32}, {65, 10}, {65, 40}, {68, 22}, {68,
  52}, {71, 34}, {78, 12}, {81, 24}}.
```

There are thus 36 elements of C_0 and the maximum degree of an element of C_0 is thus 8.

```
C_1:={{16, 14}, {19, 26}, {22, 38}, {25, 50}, {28, 62}, {29, 16}, {32,
  28}, {35, 40}, {38, 52}, {41, 64}, {42, 18}, {42, 48}, {45,
  30}, {45, 60}, {48, 42}, {51, 54}, {52, 38}, {54, 66}, {55,
  20}, {55, 50}, {58, 32}, {58, 62}, {65, 40}, {68, 22}, {68,
  52}, {71, 34}, {71, 64}, {78, 42}, {81, 24}, {81, 54}, {84, 36}}
```

There are thus 31 elements of C_1.

```
C_2:={{55, 50}, {58, 62}, {68, 52}, {71, 64}, {81, 54}, {84, 66}}
```

There are thus 6 elements of C_2. Since $|A| = 2$, these are all of the check sets.

The elements of $\tilde{S}\backslash S$ are:

```
{{24, 21}, {27, 33}, {37, 23}, {40, 35}, {50, 25}, {53, 37}}.
```

The maximum degree of an element of $\tilde{S}\backslash S$ is thus 6.

The elements of $\mathcal{B}$ are:

```
{{0, 0}, {3, 12}, {6, 24}, {9, 36}, {12, 48}, {13, 2}, {16, 14}, {19,
  26}, {22, 38}, {25, 50}, {26, 4}, {29, 16}, {32, 28}, {35, 40}, {38,
   52}, {39, 6}, {42, 18}, {52, 8}, {55, 20}, {65, 10}, {68, 22}}.
```

The maximum degree of an element of $\mathcal{B}$ is thus 6.

We can also represent the syzygies in Table 5.2.

i	m	Δ_m	$\dim_{\mathbb{K}} \tilde{H}_i(\Delta_m)$
0	{15,60}	$\{\{3\}, \{1,4\}\}$	1
0	{39,36}	{{1,3},{2,4}}	1
0	{45,30}	$\{\{1,4\}, \{2,3\}\}$	1
0	{78,12}	{{2},{1,3}}	1
1	{ 45, 60}	{{1, 3}, {1, 4}, {2, 3, 4}}	1
1	{54, 66}	{{2, 3},{1, 2, 4}, {1, 3, 4}}	1
1	{78, 42}	{{2, 4},{1, 2, 3}, {1, 3, 4}}	1
1	{84, 36}	{{1, 3}, {2, 3}, {1, 2, 4}}	1
2	{84, 66}	{{1, 2, 3},{1, 2, 4}, {1, 3, 4}, {2, 3, 4}}	1

Table 5.2: The data associated to Example 5.7.6

5.8 A simple description for $\operatorname{reg}(\mathfrak{p})$

We now use Lemma 5.7.5 to obtain a simple description for $\operatorname{reg}(\mathfrak{p})$ with respect to the standard $\mathbb{N}$-grading. We also show that this is related to the degree for which S stabilizes. Recall that $\operatorname{reg}(\mathfrak{p}) = \max\{b_i - i\}$, where b_i is the maximum ($\mathbb{N}$-graded) degree of an i-syzygy [1, Definition 1.1, p. 3].

We first recall the following notation for this Chapter. We have that S is generated by $n+1$ elements. Thus, we are taking $B = \mathbb{K}[X_0, \ldots, X_n]$ and we claim that $\operatorname{pd}\mathfrak{p} \leq r = |A| = n-1$. Indeed, we have that $\operatorname{pd}_B R = \operatorname{pd}_B \mathfrak{p} + 1$. By the graded Auslander-Buchsbaum theorem [17, Excercise 19.8, p. 489], we have that $\operatorname{pd}_B R = \operatorname{depth} B - \operatorname{depth} R$. Thus, $\operatorname{pd}_B R \leq n+1-1 = n$ (since $1 \leq \operatorname{depth} R \leq 2$) so that $\operatorname{pd}_B \mathfrak{p} \leq n-1 = r$.

We first need some preliminary results. Let $N = \max\{\deg(x) \mid x \in C_0\}$ and let M denote the elements of C_0 of degree N. Then, by definition of C_0, the elements of M will be of the form $q+a$ for some $a \in A$, $q \in \mathcal{B}$ such that $\deg(q) = N-1$ or $(d,d)+s$ for some $s \in \tilde{S}\backslash S$ such that $\deg(s) = N-2$. Note that if elements $q \in \mathcal{B}$ and $s \in \tilde{S}\backslash S$ contribute to elements of M then they are elements of maximum degree of $\mathcal{B}$ and $\tilde{S}\backslash S$ respectively. Also note that M need not have elements of both forms. More specifically, a case 1 curve will only have elements of the form $q+a$, a case 2 curve will have elements of both forms, and a case 3 curve will have elements only of the form $(d,d)+s$.

We have the following statement.

Lemma 5.8.1. *Let $N = \max\{\deg(x) \mid x \in C_0\}$ and let M denote the elements of C_0 of degree N. The following statements hold. For any element $q+a$ of M ($q \in \mathcal{B}$ such that $\deg(q) = N-1$, $a \in A$), setting $m = q + n_A$ implies that $\tilde{H}_{r-1}(\Delta_m) \neq 0$. For any element $(d,d)+s$, of M ($s \in \tilde{S}\backslash S$ such that $\deg(s) = N-2$ of M), setting $m = s + n_\Lambda$ implies that $\dim_{\mathbb{K}} \tilde{H}_r(\Delta_m) = 1$.*

Proof. We first recall that $r = |A| = n-1$. Suppose M has an element of the form $q+a$ as described in the statement. Let $m = q + a + n_{A\backslash\{a\}} = q + n_A$. We claim that $\tilde{H}_{r-1}(\Delta_m) \neq 0$. We first make some observations. Recall the long exact sequence:

$$\cdots \longrightarrow \tilde{H}_{r-1}(K_m) \longrightarrow \tilde{H}_{r-1}(\Delta_m) \longrightarrow \tilde{H}_{r-1}(\Delta_m, K_m) \longrightarrow \tilde{H}_{r-2}(K_m) \longrightarrow \cdots$$

of homology. Suppose that $\tilde{H}_{r-1}(\Delta_m, K_m) \neq 0$ and that $\tilde{H}_{r-2}(K_m) = 0$. i.e., we have the diagram:

$$\cdots \longrightarrow \tilde{H}_{r-1}(K_m) \longrightarrow \tilde{H}_{r-1}(\Delta_m) \longrightarrow \tilde{H}_{r-1}(\Delta_m, K_m) \longrightarrow 0\,.$$

Then exactness implies that $\tilde{H}_{r-1}(\Delta_m) \neq 0$.

Recall also that $\tilde{H}_{r-2}(K_m) \cong \tilde{H}_{r-1}(\bar{K}_m, K_m)$ (Section 5.2). The outline for proof is thus the following. We show that $\tilde{H}_{r-1}(\bar{K}_m, K_m) = 0$ and that $\tilde{H}_{r-1}(\Delta_m, K_m) \neq 0$.

We now show that $\tilde{H}_{r-1}(\bar{K}_m, K_m) = 0$. By Proposition 5.6.1 we have the following isomorphism:

$$\tilde{H}_{r-1}(M_m^{(r-2)}, M_m^{(r-3)}) \cong \bigoplus_{I \subseteq A, |I| = r-2, m - n_I \in S} \tilde{H}_0(T_{m-n_I}).$$

To see that $\tilde{H}_{r-1}(\bar{K}_m, K_m) = 0$, by Lemma 5.5.1, we may show that the right hand side of the above isomorphism is zero.

Suppose not. Then there exists $I \subseteq A$, $|I| = r-2$ such that $\tilde{H}_0(T_{m-n_I}) \neq 0$. Then $m - n_I = (d,d) + \tilde{s}$ for some $\tilde{s} \in \tilde{S}\backslash S$ (note that this implies that $\tilde{S}\backslash S \neq \emptyset$). By assumption, since $m = q + n_A$, $|A| = r$, and $|I| = r-2$, we have that $m - n_I = q + n_{A\backslash I}$ which has degree $N+1$. Since $q + n_{A\backslash I} = m - n_I = (d,d) + \tilde{s}$, and $\deg(q) = N-1$ it follows that $N - 1 = \deg(q) = \deg(\tilde{s})$. On the other hand, we have that $\tilde{s} + (d,d) \in C_0$ and $\deg(\tilde{s} + (d,d)) = N+1 > N$. This is a contradiction.

We now show that $\tilde{H}_{r-1}(\Delta_m, K_m) \neq 0$. For this we consider the diagram:

$$\tilde{C}.(\Delta_m, K_m) : \cdots \longrightarrow \tilde{C}_r(\Delta_m, K_m) \longrightarrow \tilde{C}_{r-1}(\Delta_m, K_m) \longrightarrow \tilde{C}_{r-2}(\Delta_m, K_m) \longrightarrow \cdots,$$

and we claim that $\tilde{C}_{r-1}(\Delta_m, K_m) \neq 0$ while both $\tilde{C}_r(\Delta_m, K_m)$ and $\tilde{C}_{r-2}(\Delta_m, K_m) = 0$.

To see that $\tilde{C}_{r-1}(\Delta_m, K_m) \neq 0$, we note that since $m - n_A \in \mathcal{B}$, A is an $r-1$-dimensional facet of Δ_m not meeting E. Thus A is a face of Δ_m not in K_m, so $\tilde{C}_{r-1}(\Delta_m, K_m) \neq 0$. On the other hand, any r-dimensional face of Δ_m must contain $r+1$ elements of Λ thus meeting E. This implies that all r-dimensional faces of Δ_m are faces of K_m, and thus $\tilde{C}_r(\Delta_m, K_m) = 0$. Finally, since $\deg(q) = \max\{\deg(x) \mid x \in \mathcal{B}\}$ we have that $q + a \notin \mathcal{B}$ for all $a \in A$. This implies that $m - n_I \in S\backslash\mathcal{B}$ for any $I \subseteq A, |I| = r-1$. Thus any $r-2$-dimensional face of Δ_m is contained in a face meeting E and thus is a face of K_m so that $\tilde{C}_{r-2}(\Delta_m, K_m) = 0$.

Thus, if M contains an element of the form $q + a$, as described in the statement, then setting $m = q + n_A$ implies that $\tilde{H}_{r-1}(\Delta_m) \neq 0$. Thus, we have constructed an $r-1$ syzygy of $\mathfrak{p}$ of degree $N + r - 1$.

Suppose now that M contains an element of the form $(d,d) + s$ for some $s \in \tilde{S}\backslash S$ such that $\deg(s) = N-2$. Let $m = (d,d) + s + n_A = s + n_\Lambda$. I claim that Δ_m is the r-sphere which will show that $\dim_{\mathbb{K}} \tilde{H}_r(\Delta_m) = 1$. By definition of $\tilde{S}$ we have that $s + e \in S$ for all $e \in E$ and I claim we also have $s + a \in S$ for all $a \in A$. Suppose not. We have the chain of sets $S \subseteq \tilde{S} \subseteq S'$, and S' is a semigroup. If $s + a \notin S$ then $s + a \in S' \backslash S$ so that for some $m, p \geq 0$, $s' = s + a + pe + me'$, is an element of $\tilde{S} \backslash S$ of degree strictly greater than s. This is a contradiction.

By assumption, we have that $m - n_\Lambda = s \notin S$. On the other hand, since $s + a \in S$ for all $a \in A$, we have that for each $L \subseteq \Lambda$, $|L| = r + 1$, $m - n_L = s + \lambda \in S$ for some $\lambda \in \Lambda$. Thus Δ_m is the r-sphere as claimed. Thanks to Leslie Roberts who noticed that Δ_m would be the r-sphere. This simplified my original argument.

Thus, if M contains an element of the form $(d, d) + s$, as described in the statement, then setting $m = s + n_\Lambda$ yields an r-syzygy of $\mathfrak{p}$ of degree $N + r$. $\square$

We have the following theorem. Our formulation of the statement and our proof is independent from [6, Theorem 16, p. 177].

Theorem 5.8.2. *With the notation as above, the following statements hold.*

$$\operatorname{reg}(\mathfrak{p}) = \max\{\deg_{\mathbb{N}}(x) \mid x \in C_0\}$$

$$= \max\{\{\max\{\textit{degree of an element of } \mathcal{B}\}+1\} \cup \{\max\{\textit{degree of an element of } \tilde{S}\backslash S\}+2\}\}$$

$$= \max\{\max\{\operatorname{ord}(x) \mid t^x \textit{ is a basis element of } \operatorname{gr}(\mathscr{S})\} + 1\} \cup$$

$$\{\max\{\operatorname{ord}(x) \mid x \textit{ is an immediately unstable element of } \operatorname{gr}(\mathscr{S})\} + 1\}\}.$$

Proof. We prove the first equality. The equality of the others follows from Lemma 5.7.5. Lemma 5.8.1 combined with Theorem 5.7.4 shows that $\operatorname{reg}(\mathfrak{p}) \geq \max\{\deg(x) \mid x \in C_0\}$. We now show that $\operatorname{reg}(\mathfrak{p}) \leq \max\{\deg_{\mathbb{N}}(x) \mid x \in C_0\}$.

Fix i such that $0 \leq i \leq \operatorname{pd} \mathfrak{p}$. As in the above discussion, we may take $0 \leq i \leq r$. Let $m \in S$ such that $\tilde{H}_i(\Delta_m) \neq 0$ and $\deg(m) = \max\{\deg(x) \mid x \in S$ and $\tilde{H}_i(\Delta_x) \neq$

$0\}$, i.e., m is an element of S corresponding to the maximum degree of an i-syzygy with respect to the standard grading. By Theorem 5.7.4 we have that $m \in C_i$. Thus we have two cases: either $m = q + n_I$ for some $I \subseteq A, |I| = i+1$, and for some $q \in \mathcal{B}$, or $m = (d,d) + s + n_{I'}$ for some $I' \subseteq A$, $|I'| = i$, and for some $s \in \tilde{S} \backslash S$.

In the first case we have that $\deg(m) - i = \deg(q) + i + 1 - i = \deg(q) + 1 \leq \max\{\deg(x) \mid x \in C_0\}$. Similarly, in the second case we have that $\deg(m) - i = \deg((d,d)+s) + i - i = \deg((d,d)+s) \leq \max\{\deg(x) \mid x \in C_0\}$. $\square$

Remark 5.8.3. Theorem 5.8.2 *does not* say that the regularity is bounded by the maximum degree of a 0-syzygy (however, it is bounded by the maximum degree of an element contained in a finite set containing all zero-syzygies).

Example 5.8.4. Continuing with Example 5.5.3, recall that $\mathscr{S} = \{7, 10, 13, 15, 24, 25\}$, we have, using *Mathematica*, that C_0 contains 127 elements. (For interest, C_1, C_2, C_3, C_4, C_5 contain $228, 267, 208, 93$, and 14 elements respectively.) The maximum degree of elements of C_0 is 10. These elements are

$$\{\{10, 240\}, \{19, 231\}, \{21, 229\}, \{24, 226\}, \{27, 223\}, \{34, 216\}\}.$$

The maximum degree of a basis element is 9 and there was only one basis element $\{9, 216\}$ of this degree. Thus, the elements of C_0 of the form $\{9, 216\} + a$ for some $a \in A$ are:

$$\{\{10, 240\}, \{19, 231\}, \{21, 229\}, \{24, 226\}, \{27, 223\}\}.$$

We have that $m = \{9, 216\} + n_A = \{65, 285\}$ and Δ_m is defined by facets:

$$\{\{2,3,5,6,7\}, \{2,3,4,5,7\}, \{2,3,4,5,6\}, \{2,3,4,6,7\}, \{2,4,5,6,7\}, \{1,2,5,6,7\},$$

$$\{1,2,4,5,7\}, \{1,2,4,6,7\}, \{1,2,3,6,7\}, \{1,3,4,5,6,7\}\}.$$

Computing homology we see that $\dim_{\mathbb{K}} \tilde{H}_4(\Delta_m) = 1$.

The maximum degree of elements of $\tilde{S}\backslash S$ is 8 and there was only one element $\{9, 191\}$. The element of C_0 of the form $(d, d) + \{9, 191\}$ is thus $\{34, 216\}$. Letting $m = \{9, 191\} + n_\Lambda = \{90, 285\}$ we have that Δ_m is the 5-sphere:

$$\{\{2,3,4,5,6,7\}, \{1,2,4,5,6,7\}, \{1,2,3,4,6,7\},$$

$$\{1,2,3,4,5,6\}, \{1,2,3,4,5,7\}, \{1,2,3,5,6,7\}, \{1,3,4,5,6,7\}\}$$

and thus $\dim_{\mathbb{K}} \tilde{H}_5(\Delta_m) = 1$.

It turns out that the minimal free resolution of $\mathfrak{p}$ is of the form (we are using nongraded notation):

$$0 \longrightarrow B^6 \longrightarrow B^{42} \longrightarrow B^{107} \longrightarrow B^{126} \longrightarrow B^{75} \longrightarrow B^{21} \longrightarrow \mathfrak{p} \longrightarrow 0\,.$$

5.9 Regularity and the stabilization of S

Recall the following notion. Let $i \geq 0$ be the smallest integer such that $t^d : \mathrm{gr}(\mathscr{S})_j \to \mathrm{gr}(\mathscr{S})_{j+1}$ is an isomorphism for all $j \geq i$. (This is equivalent to saying the same thing for s^d and $\mathrm{gr}(\hat{\mathscr{S}})$.) Then we say that S has *stabilized* in degree i.

Using Theorem 4.2.17 and Theorem 5.8.2 we have the following description of $\mathrm{reg}(\mathfrak{p})$. This description shows how the arithmetic properties of S are intrinsically related to the regularity of $\mathfrak{p}$ and how being a case 1,2 or 3 affects how regularity is attained. Essentially, we are showing that being case 1,2 or 3 is explained in the last row of the Betti diagram. Recall that $r = |A| = n - 1$.

Theorem 5.9.1. *With the notation above, let i be the integer for which S stabilizes.*

1. *If S is a case 1 curve then* $\operatorname{reg}(\mathfrak{p}) = i + 1$ *and regularity will be obtained as an* $r - 1$ *syzygy which comes from a multidegree* $q + n_A$ *for some* $q = (q_1, q_2) \in \mathcal{B}$, $\deg(q) = i$ *with* q_1 *a stable basis element of* $\operatorname{gr}(\hat{\mathscr{S}})$, *and* q_2 *a stable basis element of* $\operatorname{gr}(\mathscr{S})$. *Regularity will not be obtained as an r-syzygy.*

2. *If S is a case 2 curve then* $\operatorname{reg}(\mathfrak{p}) = i$ *and regularity will be obtained as both* $r - 1$ *and r-syzygies. Moreover, there exists an $r - 1$-syzygy attaining regularity which comes from a multidegree of the form* $m = q + n_A$ *for some* $q \in \mathcal{B}$, $\deg(q) = i - 1$. *There also exists an r-syzygy attaining regularity which comes from a multidegree of the form* $s + n_\Lambda$ *for some* $s \in \tilde{S} \backslash S$, $\deg(s) = i - 2$.

3. *If S is a case 3 curve then* $\operatorname{reg}(\mathfrak{p}) = i$ *and regularity will be obtained as an r-syzygy which comes from a multidegree of the form* $s + n_\Lambda$ *for some* $s \in \tilde{S} \backslash S$, $\deg(s) = i - 2$. *Any other t-syzygy (which may or may not be present),* $0 \leq t \leq r$, *attaining regularity will come from a multidegree of the form* $s + (d, d) + n_L$ *for some* $L \subseteq A$ *such that* $|L| = t$ *and some* $s \in \tilde{S} \backslash S$, $\deg(s) = i - 2$.

Proof. To get the claimed values for regularity we use Theorems 5.8.2 and 4.2.17. If S is a case 1 curve then, by Theorem 4.2.17, the integer i for which S stabilizes equals the maximum degree of a basis element. Moreover, all unstable elements have degree $< i$. Thus, if S is case 1, by Theorem 5.8.2, $\operatorname{reg}(\mathfrak{p}) = i + 1$. If S is case 2 then, by Theorem 4.2.17, the maximum degree of a basis element equals the maximum degree of an unstable element, which is one less than the integer i for which S stabilizes. Thus, by Theorem 5.8.2, $\operatorname{reg}(\mathfrak{p}) = i$. Similarly, if S is case 3 then by Theorem 4.2.17 the maximum degree of an unstable element is strictly greater then the maximum degree of a basis element and is one less than the integer i for which S stabilizes. Thus, by Theorem 5.8.2, $\operatorname{reg}(\mathfrak{p}) = i$.

To see how the regularity is attained, we let $N = \max\{\deg(x) \mid x \in C_0\}$. Then, by Theorem 5.8.2, in all three cases $\mathrm{reg}(\mathfrak{p}) = N$. By Theorem 5.7.4, the t-syzygies of $\mathfrak{p}$ are all contained in C_t. Thus, we fix t and describe the elements of C_t of degree $N+t$. By Theorem 5.7.4 we have

$$C_t := \{\mathcal{B} + \text{sums of } t+1 \text{ distinct elements of } A\}$$

$$\cup\{(d,d) + \ \tilde{S}\backslash S \ + \text{sums of } t \text{ distinct elements of } A\}$$

and, since $r = |A| = n-1$, $C_{n-1} = \{n_\Lambda + \tilde{S}\backslash S\}$, and that $C_t = \{\}, t \geq n$.

If S is a case 1 curve then the maximum degree of a basis element is $N-1$ and all unstable elements have degree $< N-1$. This implies, by Lemma 4.2.13, that all elements of $\tilde{S}\backslash S$ have degree $< N-2$. Thus, for a fixed $t \geq 0$, the elements of C_t of degree $N+t$ will be of the form $q+n_I$, $I \subseteq A$, $|I| = t+1$ and $\deg(q) = N-1$. Moreover, since S is case 1, Theorem 4.2.17 implies that, any basis element $q = (q_1, q_2)$ of degree $N-1$ will have t^{q_2} and s^{q_1} stable basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively. In particular, the elements of C_{r-1} of degree $N+r-1$ will be of the form $q+n_A$ where, since S is case 1, $q = (q_1, q_2)$ has degree $N-1$ and t^{q_2} and s^{q_1} are stable basis elements of $\mathrm{gr}(\mathscr{S})$ and $\mathrm{gr}(\hat{\mathscr{S}})$ respectively. We showed in Lemma 5.8.1 that elements of C_{r-1} of this form produce $r-1$-syzygies of degree $N+r-1$ and thus, attain regularity. Since there will be no elements of C_r of degree $N+r$, regularity is not attained as an r-syzygy. This completes the proof of statement 1.

If S is case 2 we argue similarly, except in this case C_{r-1} with have elements of the form $q+n_A$, $q \in \mathcal{B}$ of degree $N+r-1$ and C_r will have elements of the form $s+n_\Lambda$, $s \in \tilde{S}\backslash S$ of degree $N+r$. We showed in Lemma 5.8.1 that these elements produce $r-1$ and r-syzygies of degrees $N+r-1$ and $N+r$, respectively, and thus attain regularity. Thus, statement 2 holds.

If S is case 3 then C_r will have elements of the form $s + n_\Lambda$, $s \in \tilde{S}\backslash S$ of degree $N + r$ and, by Lemma 5.8.1, these elements produce r-syzygies of degree $N + r$ and thus attain regularity. On the other hand, since S is case 3, the maximum degree of an element of $\tilde{S}\backslash S$ is greater than or equal to the maximum degree of an element of $\mathcal{B}$ (by definition and Lemma 4.2.13) so that for a given t, the elements of C_t of degree $N + t$ will be of the form $s + (d, d) + n_I$, $s \in \tilde{S}\backslash S$, $I \subseteq A$, $|I| = t$. Thus, should these elements produce syzygies, they will attain regularity and will be of the claimed form. Thus, statement 3 holds. $\square$

Corollary 5.9.2. *The elements $S \in \mathscr{C}'$ for which* $\operatorname{reg}(\mathfrak{p})$ *is not attained in the last step of the minimal free resolution of $\mathfrak{p}$ are precisely those $S \in \mathscr{C}'$ which are case 1 and for which R is not Cohen-Macaulay.*

We now do some more examples. (Note that we are resolving $\mathfrak{p}$ as an B-module, and not $B/\mathfrak{p}$, equivalently R.)

Examples 5.9.3.

- If $\mathscr{S} = \{1, 2, 3\}$ then $\mathscr{S}$ is Cohen-Macaulay, so we are in case 1. We have that S stabilizes in degree 1, and $\operatorname{reg}(\mathfrak{p}) = 2$.

- If $\mathscr{S} = \{5, 9, 11, 20\}$ then we are in case 1. We have that S stabilizes in degree 5, and $\operatorname{reg}(\mathfrak{p}) = 6$.

- If $\mathscr{S} = \{7, 10, 13, 15, 24, 25\}$ then S stabilizes in degree 10, and we are in case 2. Thus, $\operatorname{reg}(\mathfrak{p}) = 10$. We showed in Example 5.8.4 that the regularity was obtained as both $r - 1$ and r-syzygies.

- If $\mathscr{S} = \{2, 12, 15\}$ then we are in case 3. We have that S stabilizes in degree 8 so $\operatorname{reg}(\mathfrak{p}) = 8$. We showed that regularity was obtained as an r-syzygy in Example

5.7.6. We also showed that regularity is not attained as an $r-1$-syzygy (and hence no lower syzygy.

- If $\mathscr{S} = \{1, 3, 13, 15\}$ then we have a case 3 curve which attains regularity as an $r-2$ syzygy, an $r-1$-syzygy and r-syzygy. This curve stabilizes in degree 7. The maximum degree of a basis element is 5. The maximum degree of an unstable element is 6. There are two elements of $\tilde{S}\backslash S$ of degree 5: $\{\{25, 50\}, \{37, 38\}\}$. Using [20], we obtain the Betti diagram in output 3 which shows that reg($\mathfrak{p}$) equals 7.

```
Macaulay 2, version 0.9.95
with packages: Classic, Core, Elimination, IntegralClosure, LLLBases, Parsing,
               PrimaryDecomposition, SchurRings, TangentCone

i1 : B=QQ[X_0..X_4]

o1 = B

o1 : PolynomialRing

i2 : p=monomialCurveIdeal(B,{1,3,13,15})

                          2      2     3    2     2 3    3 2   2   2    4     5
o2 = ideal (X X  - X X , X X  - X X , X  - X X , X X  - X X , X X X  - X X , X
             2 3    1 4   1 3    0 4   1    0 2   0 3    2 4   0 1 3    2 4   2
     ------------------------------------------------------------------------
        4      4    4     6      5     5    2 4   4 2    2 3
     - X X , X X  - X X , X  - X X , X X  - X X , X X  - X X X )
        0 4   1 2    0 3   3    2 4   1 3    2 4   0 3    1 2 4

o2 : Ideal of B

i3 : betti res image mingens p
```

```
             0  1  2 3
o3 = total: 9 17 12 3
         2: 1  .  . .
         3: 2  2  . .
         4: .  .  . .
         5: 4  6  2 .
         6: 2  7  6 1
         7: .  2  4 2

o3 : BettiTally
```

Considering the last row of the Betti diagram, we see that there are two 1-syzygies, four 2- syzygies and two 3-syzygies attaining regularity.

There are no 0-syzygies arising from multidegrees of ($\mathbb{N}$-graded) degree 7. There are two 1-syzygies arising from multi-degrees of degree 8: $\{\{42, 78\}, \{54, 66\}\}$ and they are of the desired form. The 2-syzygies which arise from multidegrees of degree 9 are $\{\{54, 81\}, \{56, 79\}, \{66, 69\}, \{68, 67\}\}$ and are of the desired form.

The 3-syzygies which arise from multidegrees of degree 10 are $\{\{68, 82\}, \{80, 70\}\}$ are also of the desired form.

Chapter 6

Summary and Conclusions

We now provide a summary of the thesis and make some comments about the observations made.

6.1 The total tensor resolution

The motivation behind the total tensor resolution, Theorem 3.3.1, was to show how the correspondence of Theorem 3.1.3 could be used to construct the minimal free resolutions of R and $\mathfrak{p}$ as modules over B. More specifically, given a simplicial complex Δ_m such that $\tilde{H}_i(\Delta_m) \neq 0$, we wanted to construct an i-syzygy of $\mathfrak{p}$. In the process of answering this question we noticed that Theorem 3.3.1 could be stated more generally. Moreover, by stating Theorem 3.3.1 as such, we were able to show, in Section 3.5.1, that this approach could be used to construct the minimal free resolutions of monomial ideals of B. We have not seen this approach for computing minimal free resolutions of graded B-modules done explicitly anywhere. The total tensor resolution has the advantage of always producing minimal free resolutions which is not true for other combinatorial algorithms. (See [26] for several combinatorial methods for producing (generally non-minimal) free resolutions of monomial ideals and affine semigroup rings.) We should also mention that Theorem 3.3.1 seems more useful for monomial ideals, as opposed to affine semigroup rings, since the check set containing all multidegrees of syzygies is more manageable. For example, the check sets of Theorem 5.7.4 are in practice much larger than they need to be. It would be nice to make them smaller and also to give a more conceptual description of finite check sets containing all multidegrees of syzygies.

Another observation is that in the course of proving Theorem 3.3.1 we did not directly use the assumption that B is a polynomial ring. This leads us to suspect that this theorem may hold in a more general setting i.e., finitely generated modules over an arbitrary Noetherian graded ring, or perhaps even finitely generated modules over a Noetherian local ring. Having said this it may be better, before generalizing the statement, to look for more general correspondences in the spirit of Theorem 3.1.3. Such statements would yield more situations for the hypothesis of Theorem 3.3.1 to be satisfied.

6.2 Stabilization

Chapters 4 and 5 were motivated by several factors. For example, not only did we want to understand [14], and the check sets which they propose and we describe in Theorem 5.7.4, but we also wanted to explore other applications of Theorem 3.1.3. The end result was relating the methods of [29], [30], [28] and [14] with the correspondence of Theorem 3.1.3. In particular, we introduced the notation of stabilization (Definition 4.2.15), and the definition and characterization of the cases (Definition

4.2.16 and Theorem 4.2.17). We then showed that these notions were all related to understanding how the regularity of $\mathfrak{p}$ is attained in Theorem 5.9.1. To this end, Theorem 4.2.17, Lemma 5.8.1 and Theorem 5.9.1 should be seen as the main results of this effort. Morever, the partitioning of the elements of $\mathscr{C}'$ as we have chosen to do so is also interesting, as should already be apparent from Theorem 5.9.1. We discuss other reasons in Section 6.3.

Experimental evidence suggests that the most common curves are case 2. It would be nice to prove this and explain why. It would also be interesting to be able to produce curves of each case at will. For example, experimental evidence suggests that if $x \geq 1$, $\mathscr{S}_x = \{3, 27x, 36x, 64x\}$ and $\gcd(\mathscr{S}_x) = 1$ then $\mathscr{S}_x$ defines a case 3 curve. Similarly, if $x \geq 1$, $\mathscr{S}_x = \{2, 12x, 15x\}$ and $\gcd(\mathscr{S}_x) = 1$ then $\mathscr{S}_x$ defines a case 3 curve. Experimental evidence also suggests that for case 3 curves, no 0-syzygy (of $\mathfrak{p}$) attains regularity.

Finally, it would also be interesting to understand stabilization in terms of other well studied invariants of R. For example, we saw in Theorem 5.9.1 that stabilization is not equal to the regularity of $\mathfrak{p}$ in general. Moreover, we will see other invariants for which it is not in Section 6.3.

6.3 Some comments concerning regularity

We now discuss some of our observations concerning regularity. We would first like to mention that a statement similar to Theorem 5.8.2, in slightly different language and with a more direct argument than given here, appears in [27]. The argument we give is motivated by several factors. For example, not only did we want to see how far we could push the correspondence of Theorem 3.1.3, but also, at the moment, we do not know of another way of gaining insight into how the regularity is attained and reflected in terms of the cases as we do in Theorem 5.9.1.

We would also like to mention that in [4] it is shown that if $\mathfrak{p}$ defines a curve in $\mathbb{P}^3_{\mathbb{K}}$ for which R is not Cohen-Macaulay then, in the notation of Chapters 4 and 5, $\mathrm{reg}\{\mathfrak{p}) = \max\{\deg(x) \mid x \in S'\backslash S\} + 2$. More specifically, in [4] it is shown that if $\mathfrak{p}$ defines a monomial curve in $\mathbb{P}^3_{\mathbb{K}}$ and $R \cong B/\mathfrak{p}$ then $\mathrm{reg}(R) = \max\{j \mid (H^1_{\mathfrak{m}}(R))_j \neq 0\} + 1$ so that $\mathrm{reg}(\mathfrak{p}) = \max\{j \mid (H^1_{\mathfrak{m}}(R))_j \neq 0\} + 2$. It is well known that $H^1_{\mathfrak{m}}(R) = \mathbb{K}[S'\backslash S]$, see [4, Lemma 1.1, p. 82], [32, Corollary 3.4, p. 153] or [22, Lemma 2.2] for example, so that $\mathrm{reg}(\mathfrak{p}) = \max\{\deg(x) \mid x \in S'\backslash S\}+2$. A consequence of Theorem 5.8.2 and Theorem 4.2.17 is that $\mathrm{reg}(\mathfrak{p}) = \max\{\deg(x) \mid x \in S'\backslash S\} + 2$ holds in $\mathbb{P}^n_{\mathbb{K}}$ only when R is not Cohen-Macaulay and not case 1. This observation, combined with the observation of [4] that $\mathrm{reg}(\mathfrak{p}) = \max\{\deg(x) \mid x \in S'\backslash S\} + 2$ holds in $\mathbb{P}^3_{\mathbb{K}}$, implies that there are no curves in $\mathbb{P}^3_{\mathbb{K}}$ for which R is not Cohen-Macaulay and case 1. The problem studied in [22] suggest that these are new observations. More specifically, let $\mathfrak{m} = (s^d, s^{d-m_1}t^{m_1}, \ldots, s^{m_{n-1}}t^{m_{n-1}}, t^d)$, let $\mathfrak{q} = (t^d, s^d)$ and let $r(S) = \min\{i \in \mathbb{N} \mid \mathfrak{m}^{i+1} = \mathfrak{q}\mathfrak{m}^i\}$. The number $r(S)$ is called the reduction number of $\mathfrak{m}$ with respect to $\mathfrak{q}$. In our notation, $r(S) = \min\{i \in \mathbb{N} \mid (\mathbb{K}[S]/(s^d, t^d)\mathbb{K}[S])_{i+1} = 0\}$ or, equivalently, $\max\{\deg(x) \mid x \in \mathcal{B}\}$. In [22] it is asked whether or not $\mathrm{reg}(R) = r(S)$ or, equivalently, whether or not $\mathrm{reg}(\mathfrak{p}) = r(S) + 1$. They show that this is not true, in general, by providing the counter example $\mathscr{S} = \{2, 5, 13, 14, 16, 17\}$ which turns out to be, in our language, a case 3 curve. Another consequence of Theorem 5.8.2 and Theorem 4.2.17 is that, more generally, the collection of monomial curves for which $\mathrm{reg}(R) \neq r(S)$ are those curves which are case 3. Moreover, the collection of non-Cohen-Macaulay monomial curves for which $r(S) = \max\{j \mid (H^1_{\mathfrak{m}}(R))_j \neq 0\} + 1$

are precisely those which are case 2.

Another advantage to our approach is that we are able to compute the regularity of $\mathfrak{p}$ without computing a generating set for $\mathfrak{p}$. More specifically, since both the integer for which S stabilizes and the case of S are easily computable, we conclude that the regularity of $\mathfrak{p}$ can be effectively computed by elementary means. Although we restricted our attention to monomial curves, our approach may be instructive when considering more general settings. For example, in [3] regularity of projective monomial varieties of codimension two is studied. At present we are considering the regularity of one dimensional projective monomial varieties of arbitrary codimension. As such, our methods may be instructive when considering the regularity of more general monomial varieties of codimension greater then two.

6.3.1 Explicit regularity bounds

Theorem 4.4.6 and Theorem 5.9.1 imply that if $\mathfrak{p}$ is the defining ideal of a case 1 curve corresponding to $\mathscr{S} = \{m_1, \ldots, m_n\}$ then $\mathrm{reg}(\mathfrak{p})$ is less than or equal to $m_n - n + 2$. Thus, for case 1 curves, a fairly restrictive hypothesis, we have given a combinatorial argument for the more general bound given in [21]. This also gives a partial solution to a problem posed in [10, p. 207], which asked for a combinatorial argument for the bound of [21] when $\mathfrak{p}$ defines a projective monomial curve. If $\mathfrak{p}$ is the defining ideal of a case 2 or case 3 curve then Theorem 4.4.6 and Theorem 5.9.1 imply that $\mathrm{reg}(\mathfrak{p}) \leq 2m_n - 2n + 1$. In [25, Proposition 5.5, p. 732] results from [21] are used to obtain another bound. In particular, it was shown that $\mathrm{reg}(\mathfrak{p}) \leq \max_{1 \leq i < j \leq n}\{(m_i - m_{i-1}) + (m_j - m_{j-1})\}$ ($n \geq 3$, $m_0 = 0$). (Taking $\mathscr{S} = \{1, 2, 3\}$ shows that the bounds of [21] and [25] are sharp.) The question of bounding regularity of curves is also an on-going problem of more recent interest. See [16] and [15] for example. Thus, our primary contribution to this effort is not our bounds for regularity but instead in the fact that we can compute the regularity of $\mathfrak{p}$ effectively by elementary means and also that we can explain why and how the regularity is attained. It would be interesting to explore the interplay between the methods used here and these previously mentioned accounts to see if a better bound can be obtained.

6.4 Betti numbers and the characteristic of $\mathbb{K}$

Since the property of an element of $\mathrm{gr}(\mathscr{S})$ being a basis element or an unstable element does not depend on the characteristic of the field, we conclude from Theorem 5.8.2 that the regularity of $\mathfrak{p}$ does not depend on the characteristic of the field, $\mathrm{char}\,\mathbb{K}$. This is not true for graded B-modules in general. For example, it is well known that the Betti numbers of

$$I = (X_0X_1X_2, X_1X_2X_3, X_0X_1X_4, X_0X_3X_4, X_2X_3X_4,$$

$$X_0X_2X_5, X_0X_3X_5, X_1X_3X_5, X_1X_4X_5, X_2X_4X_5),$$

which is the Stanley-Reisener ideal (see [12, Chapter 5] for a discussion of Stanley-Reisener ideals) of a triangulation of the real projective plane, as a $\mathbb{N}^6$-graded $B = \mathbb{K}[X_0, \ldots, X_5]$-module depend on $\mathrm{char}\,\mathbb{K}$. Here, we note that $\mathrm{reg}(I) = 3$ if $\mathrm{char}\,\mathbb{K} = 0$, and $\mathrm{reg}(I) = 4$ if $\mathrm{char}\,\mathbb{K} = 2$.

Having said this it remains unclear to what extent the Betti numbers of R are independent of $\mathrm{char}\,\mathbb{K}$. More specifically, in [12, Theorem 1.3, p. 188] it is shown that the Betti numbers $\beta_{i,m}$ of $R = \mathbb{K}[S]$ do not depend on $\mathrm{char}\,\mathbb{K}$ for $i = 0, 1, n-1, n$, and $n+1$. ($\beta_{n+1,m}$ will always be zero.) Since $\mathrm{Tor}^B_{i-1}(\mathbb{K}, \mathfrak{p})_m \cong \mathrm{Tor}^B_i(\mathbb{K}, R)_m$, $i \geq 1$

we conclude that the Betti numbers $\beta_{j,m}$ of $\mathfrak{p}$ do not depend on the characteristic of the field for $j = 0, n-2, n-1$ and n.

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