THE UNIVERSITY OF CALGARY

Ranges, Restrictions, Partial Maps, and Fibrations

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF COMPUTER SCIENCE

CALGARY, ALBERTA

April, 2004

Your file Votre référence
ISBN: 0-612-93373-3
Our file Notre référence
ISBN: 0-612-93373-3

Abstract

In this thesis, we study range restriction categories and their properties. Range restriction categories with split restriction idempotents are shown to be equivalent to the partial map categories of $\mathcal{M}$-stable factorization systems. The notions of a restriction fibration, a range restriction fibration, a stable meet semilattice fibration, and a range stable meet semilattice fibration are introduced and it is shown that (range) stable meet semilattice fibrations provide a bridge between the category of (range) restriction categories and the category of categories and that (range) restriction fibrations are the same as (range) restriction categories so that these fibrations provide a useful setting for studying (range) restriction categories. Finally, we construct the free range restriction structures over directed graphs using deterministic trees.

Acknowledgements

I would like to thank Professor Robin Cockett, my supervisor, for his constant guidance, support, and encouragement during my study at University of Calgary. I have benefited greatly from the courses he taught, and from numerous discussions with him. During the preparation of this thesis, he provided me with many well motivated ideas and deep insights, valuable comments and suggestions, and helpful references. Without his help, it would never have been possible to finish this thesis.

I would also like to thank Professors Clifton Cunningham, Steve Lack, and Renate Scheidler for their comments and suggestions which help to improve this thesis. As well, I am grateful to Dana Harrington, Craig Pastro, and Keith Rogers for their comments and reading.

Ruo Nan, my wife, deserves my special thanks for her love, patience, and support during my study. Thank you very much, Ruo Nan.

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Chapter 1

Introduction

Partial maps play an important role in the semantics of computer science and in the more traditional mathematical areas, such as algebraic geometry and analysis [7]. In [12], Di Paolo and Heller introduced *dominical categories* as an abstract setting to capture the notion of partiality. They used zero morphisms and near products to abstract the notion of partiality and they showed that the fundamental results of recursion theory could be obtained from these simple assumptions and the presence of a Turing object.

Robinson and Rosolini [23] noticed that the zero structure was not necessary for obtaining a notion of partiality and introduced the notion of *P-categories* (categories with a near product structure) as the basis for a more general theory of partiality. P-categories are essentially the same as Cockett's *copy categories* [5].

Cockett and Lack [7] introduced *restriction categories* as an even more general framework for working with abstract categories of partial maps. In a restriction category, the notion of partiality is captured abstractly by a single combinator $\overline{(\ )}$ and four restriction axioms (see [**R.1**], [**R.2**], [**R.3**], and [**R.4**] below). As claimed in [7], "the intuition for the combinator $\overline{f}$ is provided by thinking of the maps as programs: the restriction combinator modifies a program so that, rather than returning its output, it returns its input unchanged when it terminates." Dominical categories, P-categories, and copy categories are all restriction categories.

In this thesis, we examine categories of partial maps in which not only is the domain of the partial map abstractly defined but also the image of the partial map. This occurs frequently in practice: for example, in partial recursive functions, enumerable sets can not only be described as the domains of partial recursive functions but also as their images. We call restriction categories in which images are defined *range restriction categories* and they require, in addition to the restriction combinator, another combinator called the *range* combinator which satisfies just four axioms.

The thesis develops and studies the properties of range restriction categories.

1.1 Main Objectives of Thesis

Let X and Y be sets and $f : X \to Y$ a partial map. We define a partial map $\overline{f} : X \to X$ related to f by

$$\overline{f}(x) = \begin{cases} x & \text{if } \downarrow f(x), \\ \uparrow & \text{otherwise,} \end{cases}$$

where $\downarrow f(x)$ means that f is defined at x while $\overline{f}(x) = \uparrow$ means that $\overline{f}$ is not defined at x. Clearly, the definition of $\overline{f}$ determines the partiality of f and the following four axioms:

[R.1] $f\overline{f} = f$ for each map f,

[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[R.3] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[R.4] $\overline{g}f = f\overline{gf}$ whenever $\mathrm{cod}(f) = \mathrm{dom}(g)$, are satisfied. In [7], Cockett and Lack introduced the notion of restriction categories as a setting for working with abstract categories of partial maps: in a restriction category partiality is concentrated into this single combinator which associates to each map $f : X \to Y$ a map $\overline{f} : X \to X$ such that the above four restriction axioms [**R.1**], [**R.2**], [**R.3**], and [**R.4**] are satisfied. However, a partial map $f : X \to Y$ between sets has not only a restriction or partial structure but also a range structure. In computability theory, recursively enumerable sets can be characterized by being both domains of partial functions and ranges of recursive function: a set S is recursively enumerable if and only if S is the range of a recursive function [11]. The first objective of the thesis is to introduce the notion of *range restriction categories* to axiomize both the restriction and range structures of partial maps.

A collection of monics $\mathcal{M}$ of a category $\mathbf{C}$ is called *a stable system of monics* if it includes all isomorphisms, is closed under composition, and is stable (i.e., a pullback of an $\mathcal{M}$-map along any map is also in $\mathcal{M}$). Such a pair $(\mathbf{C}, \mathcal{M})$ is called an *$\mathcal{M}$-category,* which is essentially the same as a *domain* in [25], an *admissible system of subobjects* in [23], a *notion of partial* in [24], and a *domain structure* in [19]. $\mathcal{M}$-categories provide a categorical setting to discuss partial maps. Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, one can form its partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, which turns out to be a split restriction category (see [7]). On the other hand, for each restriction category $\mathbf{C}$, there is an $\mathcal{M}$-category $(\mathbf{D}, \mathcal{M})$ for which $\mathbf{C}$ is a full restriction subcategory of $\mathsf{Par}(\mathbf{D}, \mathcal{M})$, as shown in [7]. Cockett and Lack also showed that the construction Par is an equivalence of 2-categories between split restriction categories and $\mathcal{M}$-categories. It is natural to ask whether there is an equivalence of categories between range restriction categories and some special $\mathcal{M}$-categories. The second objective of this thesis is to answer this question by introducing the notion of $\mathcal{M}$-*stable factorization systems* and showing that the construction Par is an equivalence of 2-categories between range restriction categories with split restrictions and $\mathcal{M}$-categories, where $\mathcal{M}$ is the set of $\mathcal{M}$-maps of an $\mathcal{M}$-stable factorization systems (Theorem 2.3.8).

Fibrations are special functors important in category theory. They are connected to polymorphism, see, for example, [10], [13], and [14]. For each restriction category $\mathbf{C}$, one can form the category $\mathbf{r}(\mathbf{C})$ with (X, e_X) as objects, where e_X is a restriction idempotent over X, namely, a map $e_X : X \to X$ satisfying $\overline{e_X} = e_X$, and with maps $f : X \to Y$ such that $e_X = \overline{e_Y f e_X}$ as maps from (X, e_X) to (Y, e_Y). The obvious forgetful functor $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a very special fibration: its fibers are meet semilattices, and its inverse image functors are stable meet semilattice homomorphisms, see [7], Section 4.1 or Lemma 3.2.3 below. The third objective of the thesis is to introduce the notion of *stable meet semilattice fibrations* and to show that any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ produces a restriction category $\mathcal{S}_s(\delta_{\mathbf{X}})$ (Proposition 3.2.4) and that the functor $\mathcal{S}_s$ given by $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a left adjoint of the functor $\mathcal{R}_s$ given by $\partial_{\mathbf{C}}$ (Theorem 3.2.12). In this direction, we shall also provide the *free stable meet semilattice fibration structure* $\Delta_{\mathbf{C}}$ (Theorem 3.2.24) for each category $\mathbf{C}$. Applying the construction $\mathcal{S}_s$ to $\Delta_{\mathbf{C}}$, we shall obtain the free restriction category structure $F_r(\mathbf{C})$ over a given category $\mathbf{C}$ (Theorem 3.2.27). So stable meet semilattice fibrations provide a bridge between the category of restriction categories and the category of categories.

For each restriction category $\mathbf{C}$, the fibration $\partial_{\mathbf{C}}$ not only is a stable meet semilattice fibration but also satisfies some extra properties [**rF.1**], [**rF.2**], [**rF.3**], and [**rF**.4] (see Definition 3.3.1). In order to study the image of the category of restriction categories under $\mathcal{R}_s$, in Chapter 3, we shall also introduce the notion of *restriction fibrations* (Definition 3.3.1) and show that such restriction fibrations are the same as restriction categories (Theorem 3.3.11).

Now, one may ask whether there are similar fibrations that produce range restriction categories. In order to answer the question and to find analogous results for range restriction categories, in Chapter 4, we introduce the notion of *range stable meet semilattice fibrations* and show that such range stable meet semilattice fibrations provide range restriction structures over categories (Theorem 4.1.11). Of course, similar to restriction fibrations, we also define *range restriction fibrations* and show that those fibrations are the same as the range restriction categories (Theorem 4.2.9).

In Chapter 5, we shall provide an explicit construction of *free range restriction structures* over *directed graphs* by using *deterministic trees.* To do so, it suffices to construct the free range stable meet semilattice fibration over G^*, the path category over a given directed graph G. Hence we need the indexed category from $(G^*)^{\text{op}}$ to the category of stable meet semilattices. In order to construct that indexed category, we introduce the notions of *based directed graphs, based trees, based trees in a based directed graph*, and *deterministic trees* and get the desired free fibrations by applying *the poset collapse* to *based deterministic trees* in G.

Finally, in Chapter 6, we summarize the main results and point out some possible directions for further work.

1.2 History of Work on Partial Maps

In 1987, Di Paola and Heller [12] introduced dominical categories as an algebraic setting in which one may study partial maps and computability theory. They used zero maps and near products to approach partiality. In 1988, Robinson and Rosolini [23] noticed that one may obtain a theory of partiality with the near products alone. So they introduced P-categories (categories with near products) as the basis for their partiality theory. However, we should note here that P-categories are essentially the same as Cockett's copy categories [5]. In 1987, Carboni [4] gave a bicategorical account of partiality. In 1994, Jacobs [15] related P-categories to the semantics of weakening. In [19], [20], and [21], Mulry considered partial map classifications. In 2002, Cockett and Lack [7] abstracted the categories of partial maps as restriction categories using the single combinator $\overline{(\ )}$ and the above four simple restriction axioms [**R.1**], [**R.2**], [**R.3**], and [**R.4**]. Cockett and Lack gave a well motivated introduction to restriction categories in their paper [7], to which we refer the reader, as well as to its successors [8] and [9].

1.3 Outline of Thesis

The basic notions of category theory and the basic properties of restriction categories are assembled in Sections 1.4 and 1.5, respectively. Chapter 2 introduces the notion of range restriction categories and shows that range restriction categories with split restriction idempotents are equivalent to the partial map categories of $\mathcal{M}$-stable factorization systems. Chapter 4 explores the relation between range restriction categories and range stable meet semilattice fibrations while Chapter 3 shows the relation between restriction categories and stable meet semilattice fibrations. In Chapter 5, we provide an explicit construction of the free range restriction over directed graphs. We end this thesis by giving our conclusions in Chapter 6.

1.4 Basic Categorical Notations

In this section, we collect some basic categorical notions which we shall use.

1.4.1 Categories and Some Special Maps

A category is a directed graph with composition. More precisely, a *category* $\mathbf{C}$ consists of the following data:

- A collection of *objects*, ob$\mathbf{C}$;
- A collection of *maps*, map$\mathbf{C}$;
- Two operations assigning to each map $f \in \text{map}\mathbf{C}$ its *domain* $\text{dom}(f)$ which is an object of $\mathbf{C}$ and its *codomain* $\text{cod}(f)$ also an object of $\mathbf{C}$. We shall indicate that f has domain A and codomain B by writing $f : A \to B$;
- Maps f and g are *composable* if $\text{cod}(f) = \text{dom}(g)$. There is an operation assigning to each pair of composable maps f and g their *composition* which is a map denoted by gf such that $\text{dom}(gf) = \text{dom}(f)$ and $\text{cod}(gf) = \text{cod}(g)$. There is also an operation assigning to each object $A \in \text{ob}\mathbf{C}$ an identity map $1_A : A \to A$. These operations are required to satisfy the following axioms:

  [C.1] (identity law) if $f : A \to B$ is a map in $\mathbf{C}$ then $f1_A = f = 1_Bf$,

[C.2] (association law) if $f : A \to B$, $g : B \to C$, and $h : C \to D$ are maps in $\mathbf{C}$ then $(hg)f = h(gf)$.

A *subcategory* $\mathbf{C}'$ of a category $\mathbf{C}$ is given by any subcollections of the objects and maps of $\mathbf{C}$ which is a category under the domain, codomain, composition, and identity operations of $\mathbf{C}$.

Given a category $\mathbf{C}$, if we flip the directions of all maps in $\mathbf{C}$ then we obtain its *dual category* , denoted by $\mathbf{C}^{\mathrm{op}}$. Clearly, $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$.

Here are some examples of categories:

1. Sets and functions between them form a category **Set**.

2. Sets and partial functions between them form a category $\mathbf{Par}(\mathbf{Set}, \mathrm{Monics})$.

3. A monoid M is a category with $*$ as its object and with elements of M as its maps.

4. Groups and group homomorphisms form a category **Grp**.

5. Categories and functors form a category $\mathbf{Cat}_0$.

6. Topological spaces and continuous maps form a category **Top**.

7. Let R be a ring. Left R-modules and R-module homomorphisms form a category R-**Mod**.

8. Meet semilattices and stable meet semilattice homomorphisms between them, i.e., those maps preserving binary meets but not necessarily the top element, form a category **msLat**.

9. Posets and monotone functions form a category **Poset**.

A map $f : A \to B$ in a category $\mathbf{C}$ is *monic* if $fg_1 = fg_2$ implies $g_1 = g_2$. A map $f : A \to B$ in a category $\mathbf{C}$ is a *section* if there is a map g in $\mathbf{C}$ such that $gf = 1_A$. Dually, a map $f : A \to B$ in a category $\mathbf{C}$ is *epic* if $g_1 f = g_2 f$ implies $g_1 = g_2$ and a map $f : A \to B$ in a category $\mathbf{C}$ is a *retraction* if there is a map g in $\mathbf{C}$ such that $fg = 1_B$. A map is called an *isomorphism* if it is both a section and a retraction.

1.4.2 Functors, Natural Transformations, and Adjunctions

A *functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$, written as $F : \mathbf{C} \to \mathbf{D}$, is specified by

- an operation taking each object A in $\mathbf{C}$ to an object $F(A)$ in $\mathbf{D}$,
- an operation sending each map $f : A \to B$ in $\mathbf{C}$ to a map $F(f) : F(A) \to F(B)$ in $\mathbf{D}$,

such that

$$F(1_A) = 1_{F(A)} \text{ and } F(gf) = F(g)F(f)$$

for any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$. So, functors are *structure preserving* maps between categories.

Let $F, G : \mathbf{C} \to \mathbf{D}$ be two functors. A *natural transformation* α from F to G, written as $\alpha : F \to G$, is specified by an operation which assigns each object C of $\mathbf{C}$

a map $\alpha_C : F(C) \to G(C)$ such that for each map $f : A \to B$ in $\mathbf{C}$

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

commutes in $\mathbf{D}$. Natural transformations are maps between functors. A natural transformation α is called *a natural isomorphism*, denoted by $\alpha : F \cong G$, if each *component* α_C is an isomorphism.

An *equivalence* between categories $\mathbf{C}$ and $\mathbf{D}$ is defined to be a pair of functors $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $1_{\mathbf{C}} \cong TS$ and $1_{\mathbf{D}} \cong ST$.

An *adjunction* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $< F, G, \varphi >: \mathbf{C} \to \mathbf{D}$, where F and G are functors:

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

and φ is a function which assigns to each pair of objects $C \in \mathbf{C}, D \in \mathbf{D}$ a bijection of sets

$$\varphi = \varphi_{C,D} : \mathbf{D}(F(C), D) \cong \mathbf{C}(C, G(D))$$

which is natural in C and D. We denote it by

$$\frac{F(C) \to D}{C \to G(D)} \, .$$

By [[17], p.83, Theorem 2], each adjunction $< F, G, \varphi >: \mathbf{C} \to \mathbf{D}$ is completely determined by one of five conditions. Here we only record:

(v) *functors* F, G *and natural transformations* $\eta : 1_{\mathbf{C}} \to GF$ *and* $\varepsilon : FG \to 1_{\mathbf{D}}$ *such that* $G\varepsilon \cdot \eta G = 1_G$ *and* $\varepsilon F \cdot F\eta = 1_F$.

Hence we often denote the adjunction $< F, G, \varphi >: \mathbf{C} \to \mathbf{D}$ by $(\eta, \varepsilon) : F \dashv G : \mathbf{C} \to \mathbf{D}$ or by $< F, G, \eta, \varepsilon >: \mathbf{C} \to \mathbf{D}$.

1.4.3 2-categories, 2-Functors, and 2-Natural Transformations

A 2-*category* $\mathbf{K}$ consists of the following data:

- A collection of objects or 0-cells: $A, B, \cdots$
- A collection of maps or 1-cells: $f : A \to B, \cdots$
- A collection of 2-cells: $\alpha : f \Rightarrow g, \cdots$
- The objects and maps form a category $\mathbf{K}_0$, called the *underling category* of $\mathbf{K}$.
- For any objects A and B, the maps $f : A \to B$ and the 2-cells between them form a map-category $\mathbf{K}(A, B)$ under *vertical composition*, denoted by $\beta \circ \alpha$. The identity 2-cell on $f : A \to B$ is denoted by 1_f.
- There is an operation of *horizontal composition* of 2-cells:

$$(\beta \star \alpha : uf \to vg) = (\beta : u \to v) \star (\alpha : f \to g)$$

as displayed by

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow\alpha \\ \xrightarrow[g]{} \end{array} B \begin{array}{c} \xrightarrow{u} \\ \Downarrow\beta \\ \xrightarrow[v]{} \end{array} C \qquad = \qquad A \begin{array}{c} \xrightarrow{uf} \\ \Downarrow\beta\star\alpha \\ \xrightarrow[vg]{} \end{array} C$$

Under this operation the 2-cells form a category with identities:

$$A \begin{array}{c} \xrightarrow{1_A} \\ \Downarrow 1_{1_A} \\ \xrightarrow[1_A]{} \end{array} A.$$

- In the situation:

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow\alpha \\ \longrightarrow \\ \Downarrow\beta \\ \xrightarrow[g]{} \end{array} B \begin{array}{c} \xrightarrow{u} \\ \Downarrow\gamma \\ \longrightarrow \\ \Downarrow\delta \\ \xrightarrow[v]{} \end{array} C,$$

the following *interchange law*

$$(\delta \star \gamma) \circ (\beta \star \alpha) = (\delta \circ \beta) \star (\gamma \circ \alpha)$$

holds true and for any pair of composable 1-cells f and g,

$$1_g \star 1_f = 1_{gf}.$$

A basic example of a 2-category is **Cat**, whose objects are small categories, 1-cells are functors and 2-cells are natural transformations. Also, for any small category **C**, the slice category **Cat**/**C** is again a 2-category.

A 2-*functor* $F : \mathbf{K} \to \mathbf{L}$ between 2-categories **K** and **L** is a triple of functions sending objects, 1-cells, and 2-cells of **K** to items of the same types in **L** preserving domains, codomains, compositions, and identities.

A 2-*natural transformation* $\alpha : F \Rightarrow G$ between 2-functors $F, G : \mathbf{K} \to \mathbf{L}$ assigns to each object A of $\mathbf{K}$ a map $\alpha_A : F(A) \to G(A)$ in $\mathbf{L}$ such that for each map $f : A \to B$ in $\mathbf{K}$,

$$\alpha_B F(f) = G(f)\alpha_A$$

and for each 2-cell $\theta : f \Rightarrow g$ in $\mathbf{K}$,

$$F(A) \underset{F(g)}{\overset{F(f)}{\substack{\longrightarrow \\ \Downarrow F(\theta) \\ \longrightarrow}}} F(B) \xrightarrow{\alpha_B} G(B) \qquad = \qquad F(A) \xrightarrow{\alpha_A} G(A) \underset{G(g)}{\overset{G(f)}{\substack{\longrightarrow \\ \Downarrow G(\theta) \\ \longrightarrow}}} G(B).$$

Many categorical notions/constructions are defined *up to isomorphism.* A *pseudo-functor* is defined in such a way: if we require that those equalities in the definition of a functor hold only up to isomorphism, then we get a pseudo-functor.

1.5 Introduction to Restriction Categories

This section is devoted to the presentations of the fundamentals of Cockett-Lack's restriction theory.

1.5.1 Definitions and Basic Properties of Cockett-Lack's Restrictions

First, we recall the definition of Cockett-Lack's restriction.

A *restriction structure* on a category $\mathbf{C}$ is an assignment of a map $\overline{f} : X \to X$ to each map $f : X \to Y$ such that the four restriction axioms [**R.1**], [**R.2**], [**R.3**], and [**R.4**] are satisfied. A category with a restriction structure is called a *restriction category.*

Now, we record some basic properties of restriction categories in Lemmas 1.5.1, 1.5.2, and 1.5.3, which are Lemmas 2.1, 2.2, and 2.3 in [7], respectively.

Lemma 1.5.1 *In a restriction category,*

(*i*) $\overline{f}$ *is an idempotent for each map* f*;*

(*ii*) $\overline{f}\,\overline{gf} = \overline{gf}$ *if* $\mathrm{codom}(f) = \mathrm{dom}(g)$*;*

(*iii*) $\overline{g\overline{f}} = \overline{gf}$ *if* $\mathrm{codom}(f) = \mathrm{dom}(g)$*;*

(*iv*) $\overline{\overline{f}} = \overline{f}$ *for each map* f*;*

(*v*) $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$ *if* $\mathrm{dom}(f) = \mathrm{dom}(g)$*;*

(*vi*) *If* f *is monic then* $\overline{f} = 1$*;*

(*vii*) $f\overline{g} = f$ *implies* $\overline{f} = \overline{f}\,\overline{g}$*.*

A map f such that $f = \overline{f}$ is called a *restriction idempotent.* Restriction idempotents are precisely the maps of the form $\overline{f}$ by Lemma 1.5.1 (*iv*). A map f is called *total* if $\overline{f} = 1$.

Lemma 1.5.2 *In a restriction category,*

(*i*) *If* f *is monic, then* f *is total;*

(*ii*) *If* f *and* g *are total and* $\mathrm{codom}(f) = \mathrm{dom}(g)$ *then so is* gf*;*

(*iii*) *If* gf *is total then so is* f*;*

(*iv*) *The total maps form a subcategory.*

The subcategory of total maps of a restriction category $\mathbf{C}$ *is denoted by* $\mathsf{Total}(\mathbf{C})$*.*

A restriction idempotent $\overline{f}$ is called *split* if $\overline{f} = mr$ for some maps m and r with $rm = 1$. In such a case, m and r are called respectively *the monic part* and *the epic part* of the split restriction idempotent $\overline{f}$. Note that if $\overline{f}$ splits by m and r then $\overline{f} = \overline{r}$ as $\overline{f} = \overline{mr} = \overline{\overline{m}r} = \overline{r}$ since m is monic. A restriction structure on a category is said to be *split* if all of its restriction idempotents split. Split idempotents are determined completely by their monic parts or epic parts as shown by the following lemma:

Lemma 1.5.3 *In any restriction category:*

(*i*) *If* $rm = 1$ *and* $sm = 1$ *with* $mr = \overline{r}$ *and* $ms = \overline{s}$ *then* $r = s$*;*

(*ii*) *If* $rm = 1$ *and* $rn = 1$ *with* $mr = \overline{r}$ *and* $nr = \overline{r}$ *then* $m = n$*.*

Restriction idempotents have also the properties shown in the following lemma.

Lemma 1.5.4 *Let* $\mathbf{C}$ *be a restriction category with restriction* $\overline{(\,)}$ *and* $\mathrm{RI}_{\mathbf{C}}(X)$ *the set of all restriction idempotents over an object* X *of* $\mathbf{C}$*. Then*

(1) $\mathrm{RI}_{\mathbf{C}}(X)$ *is closed under composition;*

(2) *If* $f, g \in \mathrm{RI}_{\mathbf{C}}(X)$*, then* $gf = \overline{gf} = \overline{fg} = fg$*;*

(3) *If* $f \in \mathrm{RI}_{\mathbf{C}}(X)$*, then* $\overline{gf} = \overline{g}f$*;*

(4) $\mathrm{RI}_{\mathbf{C}}(X)$ *forms a meet semilattice.*

PROOF:

(1) If $f, g \in \mathrm{RI}_{\mathbf{C}}(X)$, then

$$\begin{aligned} \overline{gf} &= \overline{f}\,\overline{gf} \\ &= f\overline{gf} \;\; (f \in \mathrm{RI}_{\mathbf{C}}(X)) \\ &= \overline{g}f \;\; [\mathbf{R.4}] \\ &= gf \;\; (g \in \mathrm{RI}_{\mathbf{C}}(X)). \end{aligned}$$

Hence $gf \in \mathrm{RI}_{\mathbf{C}}(X)$ and therefore $\mathrm{RI}_{\mathbf{C}}(X)$ is closed under composition.

(2) $gf = \overline{g}\overline{f} = \overline{f}\overline{g} = fg$ and gf is a restriction idempotent.

(3) $\overline{gf} = \overline{g\overline{f}} = \overline{g}\overline{f}$.

(4) If we define the order $\leq$ in $\mathrm{RI}_{\mathbf{C}}(X)$ by

$$e_1 \leq e_2 \Leftrightarrow e_1 = e_1 e_2 = e_2 e_1,$$

then $\mathrm{RI}_{\mathbf{C}}(X)$ becomes a poset. Furthermore, it is easy to check that $\mathrm{RI}_{\mathbf{C}}(X)$ is a meet semilattice with the binary meet given by $e_1 \wedge e_2 = e_1 e_2$ and with the top element 1_X.

□

Let $\mathbf{C}$ be a restriction category and $f : X \to Y$ a map in $\mathbf{C}$. By Lemma 1.5.4 (4), both $\mathrm{RI}_{\mathbf{C}}(X)$ and $\mathrm{RI}_{\mathbf{C}}(Y)$ are meet semilattices. If we define $f^* : \mathrm{RI}_{\mathbf{C}}(Y) \to \mathrm{RI}_{\mathbf{C}}(X)$ by sending e in $\mathrm{RI}_{\mathbf{C}}(Y)$ to $\overline{ef}$ in $\mathrm{RI}_{\mathbf{C}}(X)$, then f^* is a *stable meet semilattice homomorphism*, which means that f^* preserves binary meets but does not necessarily preserve the top element.

Lemma 1.5.5 *Let* $\mathbf{C}$ *be a restriction category and* $f : X \to Y$ *a map in* $\mathbf{C}$*. Then* $f^* : \mathrm{RI}_{\mathbf{C}}(Y) \to \mathrm{RI}_{\mathbf{C}}(X)$*, taking* e *to* $\overline{ef}$*, is a stable meet semilattice homomorphism. Moreover,* f^* *preserves the top if and only if* f *is total.*

PROOF: For any $e_1, e_2 \in \mathrm{RI}_{\mathbf{C}}(Y)$,

$$\begin{aligned} f^*(e_1) \wedge f^*(e_2) &= \overline{e_1 f}\,\overline{e_2 f} \\ &= \overline{e_1 f \overline{e_2 f}} \text{ (by [}\mathbf{R.3}\text{])} \\ &= \overline{e_1 \overline{e_2} f} \text{ (by [}\mathbf{R.4}\text{])} \\ &= \overline{e_1 e_2 f} \\ &= f^*(e_1 e_2). \end{aligned}$$

Hence f^* preserves binary meets, as desired.

Clearly,

$$\begin{aligned} f^* \text{ preserves the top } &\Leftrightarrow 1_X = \top_{\mathrm{RI}_{\mathbf{C}}(X)} = f^*(\top_{\mathrm{RI}_{\mathbf{C}}(Y)}) = f^*(1_Y) = \overline{f} \\ &\Leftrightarrow f \text{ is total.} \end{aligned}$$

□

By Lemma 1.5.5, we have immediately:

Lemma 1.5.6 *Let* $\mathbf{C}$ *be a restriction category and let* **msLat** *be the category of all meet semilattices and stable meet semilattice homomorphisms between them. Then there is a functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{msLat}$ *sending* $f : Y \to X$ *to* $f^* : \mathrm{RI}_{\mathbf{C}}(Y) \to \mathrm{RI}_{\mathbf{C}}(X)$*.*

1.5.2 Examples of Restriction Categories

We record some examples of restriction categories as follows:

1. **Par(Set**, Monics) is a restriction category if for any partial map $f : X \to Y$, one defines the partial map $\overline{f} : X \to X$ given by

$$\overline{f}(x) = \begin{cases} x & \text{whenever } \downarrow f(x), \\ \uparrow & \text{otherwise.} \end{cases}$$

to be the restriction of f.

2. Every category is a restriction category with the restriction given by $\overline{f} = 1_X$ for any map $f : X \to Y$. The restriction is called *the trivial restriction structure.* So a restriction structure is not a property of a category but an extra structure.

3. The category displayed by

$$\overline{f} \circlearrowright \bullet \xrightarrow{f} \bullet$$

is a restriction category, where $f\overline{f} = f$ and $\overline{f}^2 = \overline{f}$.

4. If C is an object of a given restriction category $\mathbf{C}$, then the slice category $\mathbf{C}/C$ with objects all pairs (f, X), where $f : X \to C$ is a map of $\mathbf{C}$, and with maps $h : (f, X) \to (g, Y)$ those maps $h : X \to Y$ of $\mathbf{C}$ for which $gh = f$, is also a restriction category with the same restriction as $\mathbf{C}$.

In order to provide examples of one object restriction categories, we recall some definitions and properties of *inverse semigroups.*

Let S be a semigroup. $a \in S$ is called *regular* if there is $x \in S$ such that $axa = a$. A semigroup S is called *regular* if all of its elements are regular. An *inverse* of an element a is $x \in S$ such that $axa = a$ and $xax = x$. An *inverse semigroup* is a semigroup in which each element a has a unique inverse $a^{(-1)}$. Let x, y, z be elements of an inverse semigroup S. Then one has:

$$\begin{aligned} x(yz) &= (xy)z, \\ (x^{(-1)})^{(-1)} &= x, \\ (xy)^{(-1)} &= y^{(-1)}x^{(-1)}, \\ xx^{(-1)}yy^{(-1)} &= yy^{(-1)}xx^{(-1)}. \end{aligned}$$

For example, any group is an inverse semigroup. The following can be used to test when a semigroup is an inverse semigroup.

Proposition 1.5.7 *A semigroup is an inverse semigroup if and only if it is regular and any two idempotents commute.*

PROOF: See [22], p.78. □

Proposition 1.5.8 *Every inverse semigroup with an identity can be regarded as the one object restriction category with the restriction given by* $\overline{x} = x^{(-1)}x$.

PROOF: Every inverse semigroup with an identity is a monoid, so it can be regarded as a category with one object. To prove that it is a restriction category, it suffices to check the four restriction axioms.

[R.1] $x\overline{x} = xx^{(-1)}x = x$.

[R.2] $\overline{x}\,\overline{y} = x^{(-1)}xy^{(-1)}y = y^{(-1)}yx^{(-1)}x = \overline{y}\,\overline{x}$.

[R.3]

$$\begin{aligned}
\overline{y\overline{x}} &= (yx^{(-1)}x)^{(-1)}yx^{(-1)}x \\
&= x^{(-1)}(x^{(-1)})^{(-1)}y^{(-1)}yx^{(-1)}x \\
&= x^{(-1)}xy^{(-1)}yx^{(-1)}x \\
&= y^{(-1)}yx^{(-1)}xx^{(-1)}x \\
&= y^{(-1)}yx^{(-1)}x \\
&= \overline{y}\,\overline{x}.
\end{aligned}$$

[R.4] $x\overline{yx} = x(yx)^{(-1)}(yx) = xx^{(-1)}y^{(-1)}yx = y^{(-1)}yxx^{(-1)}x = y^{(-1)}yx = \overline{y}x$.

□

See [18] for relations between restriction categories and inverse semigroups.

1.5.3 Category of Restriction Categories

A functor $F : \mathbf{C} \to \mathbf{D}$ between two restriction categories is said to be a *restriction* functor if $F(\overline{f}) = \overline{F(f)}$ for any map f in $\mathbf{C}$. Restriction categories and restriction functors form a category, denoted by $\mathbf{rCat}_0$. Clearly, there is a forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ which forgets restriction structures by sending any restriction functor $F : \mathbf{C} \to \mathbf{D}$ to the functor $F : \mathbf{C} \to \mathbf{D}$.

A natural transformation between restriction functors is called a *restriction transformation* if all of its components are total. Restriction categories, restriction functors, and restriction transformations form a 2-category, called **rCat**. **rCat** has an important full 2-subcategory, comprising those objects with split restriction structures, denoted by $\mathbf{rCat}_s$.

Chapter 2

Range Restriction Categories

In this chapter, we shall introduce the notion of range restriction categories and show that partial map categories of $\mathcal{M}$-stable factorization systems provide examples of range restriction categories (Section 2.2). We shall also prove that conversely each range restriction category gives rise to a category with an $\mathcal{M}$-stable factorization system in Section 2.3. Finally, we shall prove that the 2-category of range restriction categories with split restrictions is equivalent to the 2-category of the specified factorization systems (Theorem 2.3.8).

2.1 Definition of Range Restriction Categories

Let $f : X \to Y$ be a partial map in **Par**(**Set**, Monics). We define a partial map $\widehat{f} : Y \to Y$ by

$$\widehat{f}(y) = \begin{cases} y & \text{if } \exists x \ f(x) = y, \\ \uparrow & \text{otherwise.} \end{cases}$$

Obviously the definition of $\widehat{f}$ gives the *range* of f and satisfies the following four conditions:

[RR.1] $\overline{\widehat{f}} = \widehat{f}$ for each map f,

[RR.2] $\widehat{f}f = f$ for each map f,

[RR.3] $\widehat{\overline{g}f} = \overline{g}\widehat{f}$ for all maps f, g with $\mathrm{codom}(f) = \mathrm{dom}(g)$,

[RR.4] $\widehat{g\widehat{f}} = \widehat{gf}$ for all maps f, g with $\mathrm{codom}(f) = \mathrm{dom}(g)$.

Definition 2.1.1 *A* range structure *on a restriction category* $\mathbf{C}$ *is an assignment of a map* $\widehat{f} : Y \to Y$ *in* $\mathbf{C}$ *to each map* $f : X \to Y$ *such that the four range axioms* **[RR.1]**, **[RR.2]**, **[RR.3]**, *and* **[RR.4]** *are satisfied. A restriction category with a range structure is called* a range restriction category.

2.1.1 Examples of Range Restriction Categories

1. Any category is a range restriction category with trivial restriction structure and trivial range structure given by

$$\overline{f} = 1_X \text{ and } \widehat{f} = 1_Y,$$

   for any map $f : X \to Y$.

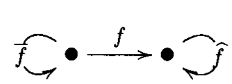
2. **Par**(**Set**, Monics) is a restriction category with restriction given by

$$\overline{f}(x) = \begin{cases} x & \text{whenever } \downarrow f(x), \\ \uparrow & \text{otherwise.} \end{cases}$$

   for each map $f : X \to Y$. It is also a range restriction category with the range structure given by

$$\widehat{f}(y) = \begin{cases} y & \text{if } \exists x \; f(x) = y, \\ \uparrow & \text{otherwise.} \end{cases}$$

3. The category displayed by



is a restriction category, where $f\overline{f} = f$, $\overline{f}^2 = \overline{f}$, $\widehat{f}f = f$, and $\widehat{f}^2 = \widehat{f}$.

Proposition 2.1.2 *Any inverse semigroup with an identity can be regarded as the one object range restriction category with the restriction and the range given by* $\overline{x} = x^{(-1)}x$ *and* $\widehat{x} = xx^{(-1)}$.

PROOF: By Proposition 1.5.8, each inverse semigroup with an identity can be regarded as the one object restriction category with the restriction given by $\overline{x} = x^{(-1)}x$. It suffices to check the four range axioms.

[RR.1] $\overline{\widehat{x}} = (xx^{(-1)})^{(-1)}xx^{(-1)} = xx^{(-1)}xx^{(-1)} = xx^{(-1)} = \widehat{x}$.

[RR.2] $\widehat{x}x = xx^{(-1)}x = x$.

[RR.3]

$$
\begin{aligned}
\widehat{\overline{y}x} &= y^{(-1)}yx(y^{(-1)}yx)^{(-1)} \\
&= y^{(-1)}yxx^{(-1)}y^{(-1)}y \\
&= y^{(-1)}yy^{(-1)}yxx^{(-1)} \\
&= y^{(-1)}yxx^{(-1)} \\
&= \overline{y}\widehat{x}.
\end{aligned}
$$

[RR.4]

$$
\begin{aligned}
\widehat{y\widehat{x}} &= yx^{(-1)}(yx^{(-1)})^{(-1)} \\
&= yx^{(-1)}xx^{(-1)}y^{(-1)} \\
&= yx^{(-1)}y^{(-1)} \\
&= yx(yx)^{(-1)} \\
&= \widehat{yx}.
\end{aligned}
$$

□

2.1.2 Some Properties of Range Restriction Categories

Some basic properties of range restriction categories are recorded in the following lemma.

Lemma 2.1.3 *In a range restriction category,*

(i) $\widehat{g}\widehat{f} = \widehat{f}\widehat{g}$ *if* codom(f) = codom(g)*;*

(ii) $\widehat{f}\overline{g} = \overline{g}\widehat{f}$ *if* dom(g) = codom(f)*;*

(iii) $\widehat{\widehat{g}f} = \widehat{g}\widehat{f}$ *if* codom(f) = codom(g)*;*

(iv) $\widehat{f} = 1$ *if f is epic. In particular,* $\widehat{1} = 1$*;*

(v) $(\widehat{f})^2 = \widehat{f}$ *for each map f;*

(vi) $\widehat{\widehat{f}} = \widehat{f}$ *for each map f;*

(vii) $\widehat{\overline{f}} = \overline{f}$ *for each map f;*

(viii) $\widehat{gf\widehat{g}} = \widehat{gf}$ *if* codom(f) = dom(g)*;*

(ix) $\widehat{\widehat{g}\widehat{f}} = \widehat{g}\widehat{f}$ *if* $\mathrm{codom}(f) = \mathrm{codom}(g)$.

PROOF:

(i) By [**RR.1**] and [**R.2**], $\widehat{g}\widehat{f} = \overline{\widehat{g}}\,\overline{\widehat{f}} = \overline{\widehat{f}}\,\overline{\widehat{g}} = \widehat{f}\widehat{g}$.

(ii) By [**RR.1**] and [**R.2**], $\widehat{f}\overline{g} = \overline{\widehat{f}}\,\overline{g} = \overline{g}\,\overline{\widehat{f}} = \overline{g}\widehat{f}$.

(iii) By [**RR.1**] and [**RR.3**], $\widehat{\widehat{g}f} = \widehat{\overline{\widehat{g}}f} = \overline{\widehat{g}}\widehat{f} = \widehat{g}\widehat{f}$.

(iv) Since f is epic, $\widehat{f}f = f$ implies $\widehat{f} = 1$.

(v) By [**RR.2**], [**RR.1**], and [**RR.3**], $\widehat{f} = \widehat{\widehat{f}f} = \widehat{\overline{\widehat{f}}f} = \overline{\widehat{f}}\widehat{f} = \widehat{f}\widehat{f} = (\widehat{f})^2$.

(vi) By [**RR.3**], $\widehat{\widehat{f}} = \widehat{\widehat{f}1} = \widehat{f}\widehat{1} = \widehat{f}$.

(vii) By [**RR.3**], $\widehat{\overline{f}} = \widehat{\overline{f}1} = \overline{f}\widehat{1} = \overline{f}$.

(viii) By *(iii)* and [**RR.2**], $\widehat{gf\widehat{g}} = \widehat{g}\widehat{gf} = \widehat{\widehat{g}gf} = \widehat{gf}$.

(ix) By [**RR.1**] and [**RR.3**], $\widehat{\widehat{g}\widehat{f}} = \widehat{\overline{\widehat{g}}\widehat{f}} = \overline{\widehat{g}}\widehat{\widehat{f}} = \widehat{g}\widehat{f}$.

□

2.1.3 2-Categories rrCat and rrCat$_s$

Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ between two restriction categories is called to be *a restriction functor* if $F(\overline{f}) = \overline{F(f)}$ and *restriction natural transformations* between two restriction functors are those natural transformations whose components are total.

Definition 2.1.4 *A functor $F : \mathbf{C} \to \mathbf{D}$ between two range restriction categories is called* a range restriction functor *if $F(\overline{f}) = \overline{F(f)}$ and $F(\widehat{f}) = \widehat{F(f)}$ for each map f in $\mathbf{C}$. A natural transformation between two range restriction functors is called a* range restriction transformation *if its components are total.*

Range restriction categories and range restriction functors form a category, denoted by $\mathbf{rrCat}_0$. There is an evident forgetful functor $U_{rr} : \mathbf{rrCat}_0 \to \mathbf{Cat}_0$, which forgets restriction and range structures. Range restriction categories, range restriction functors between them, and range restriction natural transformations form a 2-category, denoted by $\mathbf{rrCat}$. Again, there is an evident forgetful 2-functor $U_{rr} : \mathbf{rrCat} \to \mathbf{Cat}$. $\mathbf{rrCat}$ has an important full 2-subcategory, comprising those objects with split restriction, denoted by $\mathbf{rrCat}_s$.

2.2 Partial Map Categories and Range Restriction Categories

In [7], Cockett and Lack introduced the notion of $\mathcal{M}$-categories. *An $\mathcal{M}$-category* is a pair of a category $\mathbf{C}$ and a specified system of monics $\mathcal{M}$ in $\mathbf{C}$. They also used the construction Par to form a split restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and showed that Par turned out to be an equivalence of categories between the category of split restriction categories and the category of $\mathcal{M}$-categories. The objective of this section is to provide a certain class of examples of range restriction categories by showing that partial map categories with respect to the $\mathcal{M}$-maps of certain factorization systems are range restriction categories with split restriction structures.

2.2.1 Construction Par

We first summarize the notions of system of monics, $\mathcal{M}$-categories, the category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, which are given in [7].

System of Monics and $\mathcal{M}$-Categories

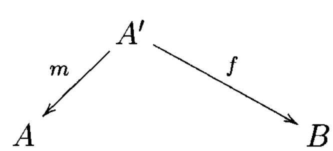
In a category, a collection $\mathcal{M}$ of monics that includes all isomorphisms and is closed under composition is called *a system of monics.* A system of monics $\mathcal{M}$ is said to be *stable* if for any $m \in \mathcal{M}$ and any $f : A \to B$ the pullback m' of m along f exists and belongs to $\mathcal{M}$. An *$\mathcal{M}$-category* is a pair $(\mathbf{C}, \mathcal{M})$, where $\mathbf{C}$ is a category and $\mathcal{M}$ is a stable system of monics in $\mathbf{C}$.

Category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$

Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, one may form the category of partial maps $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ as in [7] with:

objects: $A \in \mathbf{C}$;

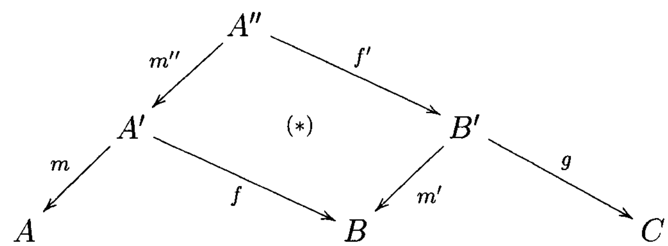
maps: a map from A to B is a pair (m, f), where $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is a map in $\mathbf{C}$:



factored out by the equivalence relation: $(m, f) \approx (m', f')$ whenever there exists an isomorphism α in $\mathbf{C}$ such that $m'\alpha = m$ and $f'\alpha = f$;

identities: $(1_A, 1_A) : A \to A$;

composition: $(m', g)(m, f) = (mm'', gf')$, where f' and m'' are given by the pull-back diagram $(*)$:



The original maps in $\mathbf{C}$ can be embedded into $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ by $f \mapsto (1, f)$ and are called *total partial maps.* In [7], Cockett and Lack proved:

Theorem 2.2.1 ([7], Proposition 3.1) *Let* $(\mathbf{C}, \mathcal{M})$ *be an* $\mathcal{M}$*-category. Then the category* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *has a split restriction given by* $\overline{(m, f)} = (m, m)$. *Furthermore, a map is total in* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *with respect to this restriction if and only if it is total as a partial map.*

2.2.2 Pullback Stability of Factorization Systems and $\mathcal{M}$StabFac

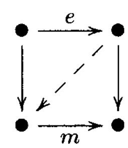
Recall that *a factorization system* on a category $\mathbf{C}$ consists of two classes $\mathcal{E}, \mathcal{M}$ of maps in $\mathbf{C}$ such that

(*i*) *every isomorphism is both in* $\mathcal{E}$ *and in* $\mathcal{M}$;

(*ii*) $\mathcal{E}$ *and* $\mathcal{M}$ *are closed under composition;*

(*iii*) *every map* f *of* $\mathbf{C}$ *factors as* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$;

(*iv*) *for each commutative square where* $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$, *there exists a unique diagonal map making both triangles commutative:*



Let $\mathcal{E}, \mathcal{M}$ be two classes of maps in $\mathbf{C}$. Then, as is well-known (see [6]), the following are equivalent:

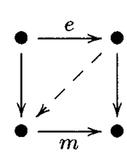
(1) $(\mathcal{E}, \mathcal{M})$ *is a factorization system on a category* $\mathbf{C}$.

(2) *The following three conditions hold true:*

(*i*) *Both* $\mathcal{M}$ *and* $\mathcal{E}$ *contain all identity maps and are closed under composition with isomorphisms on both sides;*

(*ii*) *Every map* f *of* $\mathbf{C}$ *factors as* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$;

(*iii*) *For each commutative square where* $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$, *there exists a unique diagonal map making both triangles commutative:*

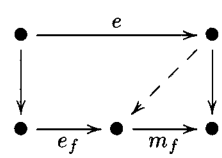


(3) *The following two conditions hold true:*

(*i*) $\mathcal{E}$ *contains all isomorphisms and is closed under composition;*

(*ii*) *Maximal $\mathcal{E}$-factorization property: each map f can be factored as $f = m_f e_f$ with $e_f \in \mathcal{E}$, which is maximal in the following sense:*

for each commutative square with $e, e_f \in \mathcal{E}$, there is a unique map making



commute.

Furthermore, under the above condition (3), *if* $\mathcal{M} = \{f \mid e_f \text{ is isomorphic}\}$, *then* $(\mathcal{E}, \mathcal{M})$ *forms the factorization system determined by the condition* (3).

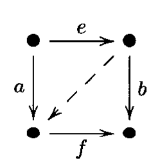
A factorization system has the following properties.

Proposition 2.2.2 *Let* $(\mathcal{E}, \mathcal{M})$ *be a factorization system on a category* $\mathbf{C}$. *Then*

(*i*) $f \in \mathcal{E} \cap \mathcal{M}$ *implies* f *is an isomorphism;*

(*ii*) *The factorization* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$, *of a map* f, *is unique up to an isomorphism;*

(*iii*) $f \in \mathcal{M}$ *if and only if for every commutative square*



with $e \in \mathcal{E}$ *there is a unique diagonal map making both triangles commutative;*

(*iv*) $fg \in \mathcal{M}$ *and* $f \in \mathcal{M}$ *implies* $g \in \mathcal{M}$;

(*v*) *Pullbacks of $\mathcal{M}$-maps are $\mathcal{M}$-maps. That is if*

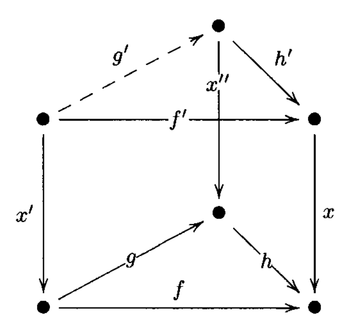
$$\begin{array}{ccc} \bullet & \xrightarrow{n} & \bullet \\ a \downarrow & & \downarrow b \\ \bullet & \xrightarrow[m]{} & \bullet \end{array}$$

is a pullback diagram and $m \in \mathcal{M}$, then $n \in \mathcal{M}$.

The dual properties of (*iii*), (*iv*), (*v*) *are valid for maps in $\mathcal{E}$.*

PROOF: See [3]. □

Now, we consider the following diagram:

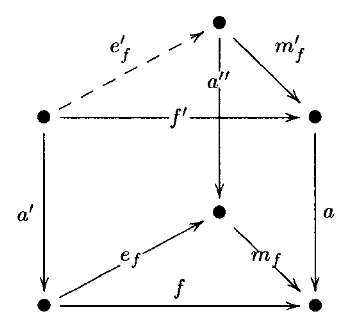


in which f' and h' are pullbacks of f and h along x, respectively. If $f = hg$, then, by easy diagram chasing, there is a unique map g' such that $f' = h'g'$ and $x''g' = gx'$ and so g' is the pullback of g along x''. We say that the triangle $f' = h'g'$ is a *pullback* of the triangle $f = hg$ along x.

Definition of Pullback Stable Factorization Systems

Definition 2.2.3 *Let* **C** *be a category and let $\mathcal{A}$ be a set of maps in* **C**, *along which pullbacks exist. A factorization system $(\mathcal{E}, \mathcal{M})$ of* **C** *is said to be* pullback stable along $\mathcal{A}$-maps *if for any* $a \in \mathcal{A}$ *and any* $(\mathcal{E}, \mathcal{M})$*-factorization* $f = m_f e_f$, $f' = m'_f e'_f$ *is a pullback of* $f = m_f e_f$ *along* a*, then* $f' = m'_f e'_f$ *is the* $(\mathcal{E}, \mathcal{M})$*-factorization of* f'*.*

Clearly, a factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathbf{C}$ is *pullback stable along* $\mathcal{A}$*-maps* in the case that for any $(\mathcal{E}, \mathcal{M})$-factorization $f = m_f e_f$ in $\mathbf{C}$ and any $\mathcal{A}$-map a, if $f' = m'_f e'_f$ is a pullback of $f = m_f e_f$ along a, then $f' = m'_f e'_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f':



Examples

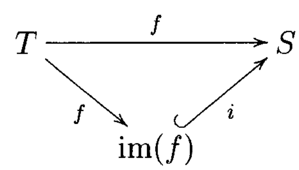
1. Let $\mathbf{Top}_{\text{open}}$ be the subcategory of $\mathbf{Top}$ with open functions as its maps. Consider

   $$\mathcal{E} = \{\text{surjective open functions}\},$$

   $$\mathcal{M} = \{\text{injective open functions}\}.$$

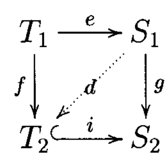
   Clearly, both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms.

   For any open function $f : T \to S$, $f : T \to \text{im}(f)$ is a surjective open function and $i : \mathrm{im}(f) \hookrightarrow S$ is an injective open function. So



gives the $(\mathcal{E}, \mathcal{M})$-factorization of f.

For any commutative digram

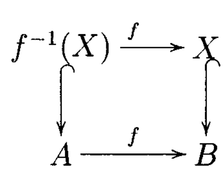


where $e \in \mathcal{E}, i \in \mathcal{M}$, since

$$g(S_1) = ge(T_1) = if(T_1) \subseteq i(Y_1),$$

there is a unique diagonal map d given by g, which makes both triangles commutative. Hence $\mathbf{Top}_{\mathrm{open}}$ admits $(\mathcal{E}, \mathcal{M})$-factorization system.

If $f : A \to B$ is an $\mathcal{E}$-map and $X \subseteq B$ is open, then $f^{-1}(X)$ is an open subset of A, and so open sets of $f^{-1}(X)$ and X are given by those of A and B, respectively. Hence $f : f^{-1}(X) \to X$ is also an $\mathcal{E}$-map. Note that



is a pullback diagram. Then the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbf{Top}_{\text{open}}$ is pullback stable.

2. Recall that a category is *regular* if each map has a kernel pair and each kernel pair has a coequalizer and if regular epics are pullback stable. The algebraic and monadic categories over **Set**, including Ω-algebras, are regular [2]. Any regular category admits the (RegEpi, Mon)-factorization system which is pullback stable [2].

2-Category $\mathcal{M}$StabFac

Suppose that **$\mathcal{M}$StabFac** is with

objects: $\mathcal{M}$-stable factorization systems $(\mathbf{C}, \mathcal{E}, \mathcal{M})$, where $\mathbf{C}$ is a category such that $\mathbf{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorization system which is pullback stable along $\mathcal{M}$-maps with $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$, and $\mathbf{C}$ has pullbacks along $\mathcal{M}$-maps;

maps: $(\mathcal{E}, \mathcal{M})$-functors. A $(\mathcal{E}, \mathcal{M})$-functor $F : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ is a functor $F : \mathbf{C} \to \mathbf{C}'$ such that $F(\mathcal{E}) \subseteq \mathcal{E}'$, $F(\mathcal{M}) \subseteq \mathcal{M}'$, and F preserves pullbacks along $\mathcal{M}$-maps;

composition: as the composition of functors;

identities: $1_{(\mathbf{C},\mathcal{E},\mathcal{M})} = 1_{\mathbf{C}}$;

2-cells: $\mathcal{M}$-cartesian natural transformations. A natural transformation $\alpha : F \to G$ between $(\mathcal{E}, \mathcal{M})$-functors: $F, G : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ is *$\mathcal{M}$-cartesian* if for each $m : A \to B$ in $\mathcal{M}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(m)}\downarrow & & \downarrow{\scriptstyle G(m)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

is a pullback diagram.

Then, $\mathcal{M}$**StabFac** is a 2-category.

When is a Factorization System Pullback Stable?

Lemma 2.2.4 *If pullbacks along $\mathcal{M}$-maps exist, then a factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathbf{C}$ is pullback stable along $\mathcal{M}$-maps if and only if $\mathcal{E}$ is pullback stable along $\mathcal{M}$-maps.*

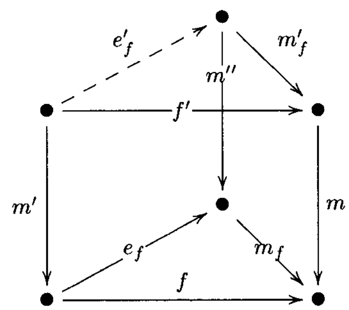
PROOF: $(\Rightarrow)$ For any $e \in \mathcal{E}$ and any $m \in \mathcal{M}$, if

$$\begin{array}{ccc} \bullet & \xrightarrow{e'} & \bullet \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ \bullet & \xrightarrow{e} & \bullet \end{array}$$

is a pullback diagram, then $e' = 1e'$ is the pullback of the $(\mathcal{E}, \mathcal{M})$-factorization $e = 1e$ of e and so is the $(\mathcal{E}, \mathcal{M})$-factorization of e'. Hence $e' \in \mathcal{E}$.

$(\Leftarrow)$ For any $m \in \mathcal{M}$, assume that $f = m_f e_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f and that f' and m'_f are pullbacks of f and m_f along m, respectively. Then there exists a unique map e'_f such that $f' = m'_f e'_f$ and $m'' e'_f = e_f m'$ and e'_f is the pullback of e_f

along $m'' \in \mathcal{M}$:



and so that $e'_f \in \mathcal{E}$ by hypothesis. Since $\mathcal{M}$-maps are pullback stable, $m'_f \in \mathcal{M}$. So $f' = m'_f e'_f$ is $(\mathcal{E}, \mathcal{M})$-factorization of f'. □

2.2.3 Factorization System Implies Range Restriction

If $\mathbf{C}$ is a category with a specified $(\mathcal{E}, \mathcal{M})$-factorization system which is stable over $\mathcal{M}$-maps, then $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ becomes a range restriction category as shown in Theorem 2.2.5 below. So we can construct a range restriction category whenever we have an $(\mathcal{E}, \mathcal{M})$-factorization system which is $\mathcal{M}$-stable and pullbacks along $\mathcal{M}$-maps exist as shown in the following theorem.

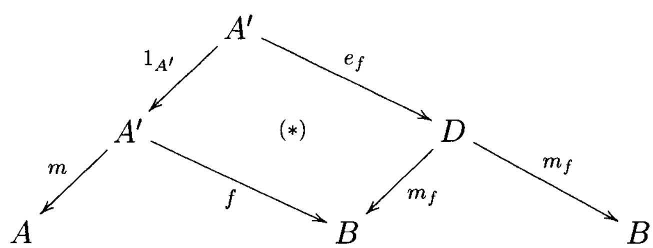
Theorem 2.2.5 *Let* $\mathbf{C}$ *be a category with an* $(\mathcal{E}, \mathcal{M})$*-factorization system which is pullback stable along* $\mathcal{M}$*-maps with* $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$*. If* $\mathbf{C}$ *has pullbacks along* $\mathcal{M}$*-maps, then* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *is a range restriction category with the split restriction structure given by* $\overline{(m, f)} = (m, m)$ *and the range structure given by* $\widehat{(m, f)} = (m_f, m_f)$*, where* m_f *is determined by the* $(\mathcal{E}, \mathcal{M})$*-factorization of* f*:* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$*. Furthermore, a map is total in* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *if and only if it is total as a partial map.*

PROOF: Since $\mathbf{C}$ admits an $(\mathcal{E},\mathcal{M})$-factorization system with $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$, $(\mathbf{C},\mathcal{M})$ is an $\mathcal{M}$-category. By Theorem 2.2.1, $\mathsf{Par}(\mathbf{C},\mathcal{M})$ is a restriction category with the split restriction given by $\overline{(m,f)} = (m,m)$ and a map is total in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ with respect to this restriction if and only if it is total as a partial map. In order to prove that $\mathsf{Par}(\mathbf{C},\mathcal{M})$ is a range restriction category with the range structure given by $\widehat{(m,f)} = (m_f, m_f)$, where m_f is given by the $(\mathcal{E},\mathcal{M})$-factorization of f: $f = m_f e_f$, it suffices to check the four range axioms.

Let $(m,f) : A \to B$ and $(n,g) : B \to C$ be maps in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ and let $f = m_f e_f$ be the $(\mathcal{E},\mathcal{M})$-factorization of f.

[RR.1] $\overline{\widehat{(m,f)}} = \overline{(m_f,m_f)} = (m_f,m_f) = \widehat{(m,f)}$.

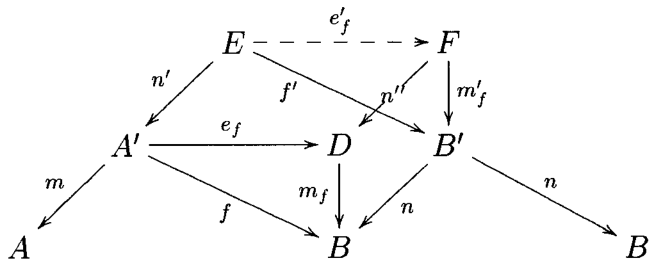
[RR.2] $\widehat{(m,f)}(m,f) = (m_f,m_f)(m,f) = (m,f)$ since the following (*) is a pullback diagram:



[RR.3] Let (n',f') be the pullback of (n,f) and (n'',m'_f) the pullback of (n,m_f). Then there is a unique map $e'_f : D \to D'$ in $\mathbf{C}$ such that

$$m'_f e'_f = f' \text{ and } n'' e'_f = e_f n'$$

and so (n', e'_f) is a pullback of (n'', e_f):



Hence, by hypothesis, $f' = m'_f e'_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f' and therefore

$$nf' = (nm'_f)e'_f$$

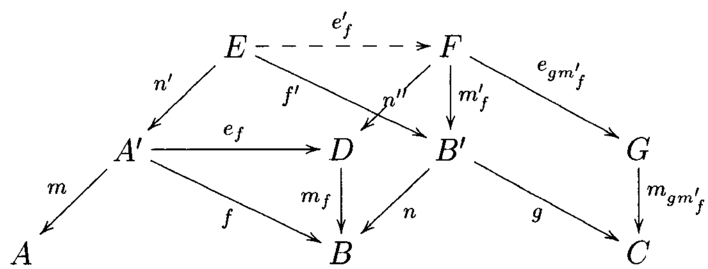
is the $(\mathcal{E}, \mathcal{M})$-factorization of nf' since $nm'_f \in \mathcal{M}$ and $e'_f \in \mathcal{E}$. Thus,

$$\begin{aligned}
\widehat{\overline{(n,g)}(m,f)} &= \widehat{(n,n)(m,f)} \\
&= \widehat{(mn', nf')} \\
&= (m_{nf'}, m_{nf'}) \\
&= (nm'_f, nm'_f) \text{ (since } nf' = (nm'_f)e'_f) \\
&= (n,n)(m_f, m_f) \text{ (since } (n'', m'_f) \text{ is the pullback of } (n, m_f)) \\
&= \overline{(n,g)}\widehat{(m,f)}.
\end{aligned}$$

[RR.4] Suppose that (f', n') and (m'_f, n'') are pullbacks of (f, n) and (m_f, n), respectively. Again, there is a unique map $e'_f : E \to F$ such that

$$m'_f e'_f = f' \text{ and } n'' e'_f = e_f n'$$

and so (n', e'_f) is a pullback of (n'', e_f):



If $gm'_f = m_{gm'_f} e_{gm'_f}$ is the $(\mathcal{E}, \mathcal{M})$-factorization of gm'_f, then

$$gf' = g(m'_f e'_f) = m_{gm'_f}(e_{gm'_f} e'_f)$$

is the $(\mathcal{E}, \mathcal{M})$-factorization of gf'. Thus,

$$\begin{aligned}
\widehat{(n,g)\widehat{(m,f)}} &= (n,g)\widehat{(m_f, m_f)} \\
&= \widehat{(m_f n'', gm'_f)} \\
&= (m_{gm'_f}, m_{gm'_f}) \\
&= (m_{gf'}, m_{gf'}) \\
&= \widehat{(m_{gf'}, m_{gf'})} \\
&= (n,g)\widehat{(m,f)},
\end{aligned}$$

as desired.

Hence, $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a range restriction category. $\square$

If $F : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ is a map in $\mathcal{M}$**StabFac**, then $F(\mathcal{E}) \subseteq \mathcal{E}'$, $F(\mathcal{M}) \subseteq \mathcal{M}'$, and F preserves pullbacks along $\mathcal{M}$-maps. Hence $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to$

$\mathsf{Par}(\mathbf{C}', \mathcal{M}')$ given by

$$\begin{array}{ccccc} & A' & & & & F(A') & \\ {}^{m}\swarrow & & \searrow^{f} & \mapsto & {}^{F(m)}\swarrow & & \searrow^{F(f)} \\ A & & B & & F(A) & & F(B) \end{array}$$

is a range restriction functor. So we have the functor $\mathsf{Par} : \mathcal{M}\mathbf{StabFac} \to \mathbf{rrCat}$ given by

$$\begin{array}{ccc} (\mathbf{C}, \mathcal{E}, \mathcal{M}) & \mapsto & \mathsf{Par}(\mathbf{C}, \mathcal{M}) \\ {\scriptstyle F}\downarrow & \mapsto & \downarrow{\scriptstyle \mathsf{Par}(F)} \\ (\mathbf{C}', \mathcal{E}', \mathcal{M}') & \mapsto & \mathsf{Par}(\mathbf{C}', \mathcal{M}') \end{array}$$

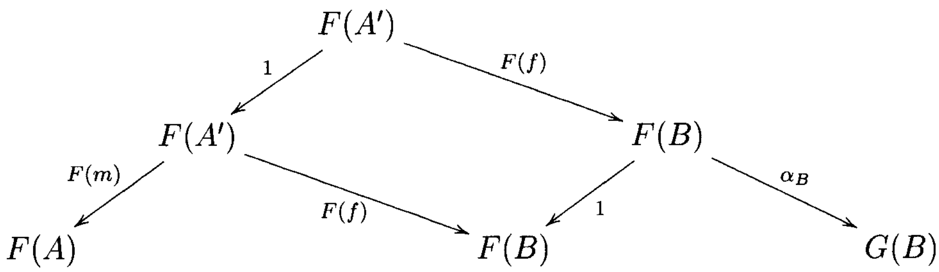
Furthermore, it is a 2-functor as shown by:

Proposition 2.2.6 *There is a 2-functor* $\mathsf{Par} : \mathcal{M}\mathbf{StabFac} \to \mathbf{rrCat}$ *taking* $F : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ *to* $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{C}', \mathcal{M}')$.

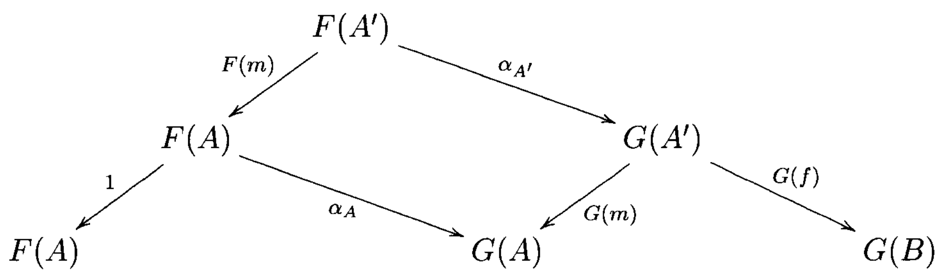
PROOF: In order to define a 2-functor, we should also point out the assignments on the 2-cells. Given $F, G : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ and an $\mathcal{M}$-cartesian natural transformation $\alpha : F \to G$, define $\mathsf{Par}(\alpha)_A : \mathsf{Par}(F)(A) \to \mathsf{Par}(G)(A)$ to be the total map $(1_A, \alpha_A) : F(A) \to G(A)$. We must check the naturality condition which says that

$$\begin{array}{ccc} F(A) & \xrightarrow{(1,\alpha_A)} & G(A) \\ {\scriptstyle (F(m),F(f))}\downarrow & & \downarrow{\scriptstyle (G(m),G(f))} \\ F(B) & \xrightarrow{(1,\alpha_B)} & G(B) \end{array}$$

and amounts to checking that



and



give the same composition. This follows immediately by the $\mathcal{M}$-cartesianness of α and the naturality of α.

All the data for a 2-functor $\mathsf{Par} : \mathcal{M}\mathbf{StabFac} \to \mathbf{rrCat}$ are now ready. Checking various functoriality conditions remains but is straightforward. $\square$

2.3 The Completeness of Range Restriction Categories

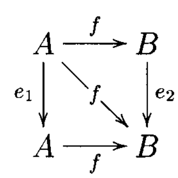
In [7], Cockett and Lack provided the completeness of restriction categories as a formulation of partial maps by showing that each restriction category is a restriction subcategory of $\mathbf{Par}(\mathbf{D}, \mathcal{M})$ for some $\mathcal{M}$-category $(\mathbf{D}, \mathcal{M})$ and that $\mathbf{rCat}_s$ is equivalent to the category of $\mathcal{M}$-categories. In this section, we shall show that in fact each range restriction category is a range restriction subcategory of $\mathbf{Par}(\mathbf{D}, \mathcal{M})$ for some category $\mathbf{D}$ with an $\mathcal{M}$-stable factorization system and that the category $\mathbf{rrCat}_s$ of range restriction categories with split restriction is equivalent to $\mathcal{M}$**StabFac**.

2.3.1 Construction Split_E

In order to prove a converse of Theorem 2.2.5, we assume that $\mathbf{C}$ is a range restriction category. However, this category does not have a class of $\mathcal{M}$-maps nor need it have any pullbacks. In order to introduce the desired structure, we form $\mathsf{Split}_E(\mathbf{C})$ as follows:

objects: restriction idempotents of $\mathbf{C}$;

maps: a map f from $(e_1 : A \to A)$ to $(e_2 : B \to B)$ is given by a map $f : A \to B$ in $\mathbf{C}$ such that both triangles in the diagram



are commutative;

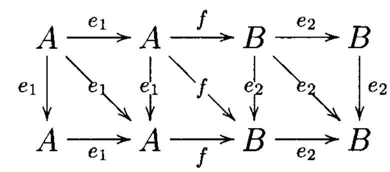
composition: as in $\mathbf{C}$;

identities: $1_e = e$ for any object e of $\mathsf{Split}_E(\mathbf{C})$.

If $f : e_1 \to e_2$ and $g : e_2 \to e_3$ are maps in $\mathsf{Split}_E(\mathbf{C})$, then

$$e_3(gf) = (e_3g)f = gf \text{ and } (gf)e_1 = g(fe_1) = gf,$$

and so gf is a map from e_1 to e_3 in $\mathsf{Split}_E(\mathbf{C})$. Hence the composition is well-defined. Obviously, the composition is associative. Since $ee = e = ee$, clearly e is a map from e to e so that identities are well-defined. For any map $f : e_1 \to e_2$ in $\mathsf{Split}_E(\mathbf{C})$, since all triangles of the diagram



are commutative,

$$f1_{e_1} = fe_1 = f = e_2 f = 1_{e_2} f.$$

Therefore, $\mathsf{Split}_E(\mathbf{C})$ is indeed a category. Furthermore, $\mathsf{Split}_E(\mathbf{C})$ is a *range restriction category* when we define its restriction and range structures by the range and restriction structures in $\mathbf{C}$. To show this, it suffices to show that $\overline{f} : e_1 \to e_1$ and $\widehat{f} : e_2 \to e_2$ are maps of $\mathsf{Split}_E(\mathbf{C})$.

Lemma 2.3.1 *If* $f : e_1 \to e_2$ *is a map of* $\mathsf{Split}_E(\mathbf{C})$, *then so are* $\overline{f} : e_1 \to e_1$ *and* $\widehat{f} : e_2 \to e_2$.

PROOF: Since

$$\overline{f}e_1 = \overline{f}\,\overline{e_1} = \overline{f\overline{e_1}} = \overline{fe_1} = \overline{f},$$

$$e_1\overline{f} = \overline{f}e_1 = \overline{f},$$

$$e_2\widehat{f} = \overline{e_2}\widehat{f} = \widehat{\overline{e_2}f} = \widehat{e_2 f} = \widehat{f},$$

and

$$e_2\widehat{f} = \widehat{f}e_2 = f,$$

all triangles in

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}} & A \\ {\scriptstyle e_1}\downarrow & \overset{\overline{f}}{\searrow} & \downarrow{\scriptstyle e_1} \\ A & \xrightarrow[\overline{f}]{} & A, \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\widehat{f}} & B \\ {\scriptstyle e_2}\downarrow & \overset{\widehat{f}}{\searrow} & \downarrow{\scriptstyle e_2} \\ B & \xrightarrow[\widehat{f}]{} & B \end{array}$$

are commutative. Hence $\overline{f} : e_1 \to e_1$ and $\widehat{f} : e_2 \to e_2$ are maps in $\mathsf{Split}_E(\mathbf{C})$, as desired. □

Therefore, $\mathsf{Split}_E(\mathbf{C})$ *is a range restriction category.* But we have more:

Proposition 2.3.2 *If* $\mathbf{C}$ *is a range restriction category, then so is* $\mathsf{Split}_E(\mathbf{C})$*, but with a split restriction structure given by the restriction in the category* $\mathbf{C}$.

PROOF: It remains to prove that the restriction structure of $\mathsf{Split}_E(\mathbf{C})$ is split. For any restriction idempotent $\overline{f}$ given by $\mathbf{C}$-map $f : (e_1 : A \to A) \to (e_2 : B \to B)$, since all triangles in

$$\begin{array}{ccccc} A & \xrightarrow{\overline{f}} & A & \xrightarrow{\overline{f}} & A \\ {\scriptstyle e_1}\downarrow & \overset{\overline{f}}{\searrow} & \downarrow{\scriptstyle \overline{f}} & \overset{\overline{f}}{\searrow} & \downarrow{\scriptstyle e_1} \\ A & \xrightarrow[\overline{f}]{} & A & \xrightarrow[\overline{f}]{} & A \end{array}$$

are commutative, $\overline{f} : e_1 \to \overline{f}$ and $\overline{f} : \overline{f} \to e_1$ are maps of $\mathsf{Split}_E(\mathbf{C})$. Note that

$$\overline{f}\,\overline{f} = \overline{f} = 1_{\overline{f}}$$

in $\mathsf{Split}_E(\mathbf{C})$. Hence $\overline{f}$ is a split restriction, as desired. □

2.3.2 Range Restriction Implies Factorization System

Let $\mathbf{D}$ be a range restriction category with split restriction. Then we consider the following two classes of maps:

$$\mathcal{E}_{\mathbf{D}} = \{f : X \to Y \,\text{in}\, \mathsf{Total}(\mathbf{D}) \,|\, \widehat{f} = 1_Y\}$$

and

$$\mathcal{M}_{\mathbf{D}} = \{m : X \to Y \,\text{in}\, \mathsf{Total}(\mathbf{D}) \,|\, \exists\, r : Y \to X \text{ in } \mathbf{D},\ rm = 1_X \text{ and } \overline{r} = mr\}.$$

We have:

Theorem 2.3.3 *If* $\mathbf{D}$ *is a range restriction category with split restriction, then* $\mathsf{Total}(\mathbf{D})$ *admits the* $(\mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$*-factorization system which is pullback stable along* $\mathcal{M}_{\mathbf{D}}$*-maps and* $\mathsf{Total}(\mathbf{D})$ *has pullbacks along* $\mathcal{M}_{\mathbf{D}}$*-maps, where* $\mathcal{E}_{\mathbf{D}}$ *and* $\mathcal{M}_{\mathbf{D}}$ *are as above with*

$$\mathcal{M}_{\mathbf{D}} \subseteq \{\text{monics in } \mathsf{Total}(\mathbf{D})\}.$$

The proof of Theorem 2.3.3 is given by the following three lemmas:

Lemma 2.3.4 *If* $\mathbf{D}$ *is a range restriction category with split restriction, then the category* $\mathsf{Total}(\mathbf{D})$ *admits the* $(\mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$*-factorization system with*

$$\mathcal{M}_{\mathbf{D}} \subseteq \{\text{monics in } \mathsf{Total}(\mathbf{D})\}.$$

PROOF: Clearly, every isomorphism is in $\mathcal{E}_{\mathbf{D}}$ and $\mathcal{M}_{\mathbf{D}} \subseteq \{\text{monics in } \mathsf{Total}\mathbf{D}\}$. The composition closedness of $\mathcal{E}_{\mathbf{D}}$ is clear since restriction idempotents are closed under composition and $\widehat{f} = \overline{\widehat{f}}$ for any map f in $\mathbf{D}$.

For any map $f : X \to Y$ in $\mathsf{Total}(\mathbf{D})$, since $\widehat{f} : Y \to Y$ is a split restriction idempotent, we can write $\widehat{f} = m_f r_f$ for some maps $r_f : Y \to Z$ and $m_f : Z \to Y$ with $r_f m_f = 1_Z$. Then $f = \widehat{f} f = m_f(r_f f)$. Clearly, $m_f \in \mathcal{M}_{\mathbf{D}}$. To prove that $m_f(r_f f)$ is the $(\mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$-factorization of f, we need to check that $r_f f \in \mathcal{E}_{\mathbf{D}}$. But, it is easy since

$$
\begin{aligned}
\overline{r_f f} &= \overline{\overline{m_f} r_f f} \text{ (since } m_f \text{ is monic)} \\
&= \overline{m_f r_f f} \text{ (by Lemma 1.5.1 } (iii)) \\
&= \overline{\widehat{f} f} \\
&= \overline{f} \text{ (since } f \in \mathsf{Total}(\mathbf{D})) \\
&= 1_X,
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{r_f f} &= \widehat{r_f \widehat{f}} \text{ (by [}\mathbf{RR.4}\text{])} \\
&= \widehat{r_f m_f r_f} \\
&= \widehat{r_f} \\
&= 1_Z \text{ (since } r_f \text{ is epic).}
\end{aligned}
$$

For each commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ C & \xrightarrow{m} & D \end{array}$$

in $\mathsf{Total}(\mathbf{D})$, where $e \in \mathcal{E}_{\mathbf{D}}$ and $m \in \mathcal{M}_{\mathbf{D}}$, we assume that $r : D \to C$ is a map in $\mathbf{D}$ such that $rm = 1_C$ and $\overline{r} = mr$. Then

$$\overline{r}\,\widehat{mx} = \widehat{\overline{mr}mx} = \widehat{mrmx} = \widehat{mx},$$

and so

$$\widehat{mx} \leq \overline{r}.$$

Hence

$$mr = \overline{r} \geq \widehat{mx} = \widehat{ye} = \widehat{y\widehat{e}} = \widehat{y},$$

and therefore $mr\widehat{y} = \widehat{y}$. It follows that

$$m(ry) = mr\widehat{y}y = \widehat{y}y = y.$$

Clearly,

$$(ry)e = r(mx) = (rm)x = x.$$

Since $y = mry$ is total, by Lemma 1.5.2 (*iii*), ry is total. Hence, m is monic implies that there exists a unique diagonal map $ry : B \to C$ in $\mathsf{Total}(\mathbf{D})$ making both triangles in

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle x}\downarrow & \swarrow{\scriptstyle ry} & \downarrow{\scriptstyle y} \\ C & \xrightarrow{m} & D \end{array}$$

commute. So $\mathsf{Total}(\mathbf{D})$ has the $(\mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$-factorization system. $\square$

Lemma 2.3.5 *Pullbacks of a total map along $\mathcal{M}_{\mathbf{D}}$-maps exist in $\mathsf{Total}(\mathbf{D})$. More precisely, suppose that $m \in \mathcal{M}_{\mathbf{D}}$ such that $\overline{r} = mr, rm = 1$ and f is a total map. If $\overline{rf} = m'r'$ with $r'm' = 1$ and $f' = rfm'$, then (m', f') is a pullback of (m, f).*

PROOF: Since r' is epic and

$$mf'r' = m(rfm')r' = mrf\overline{rf} = \overline{r}f = f\overline{rf} = fm'r',$$

we have $mf' = fm'$. Clearly, $\overline{m'} = 1$ and so m' is in $\mathsf{Total}(\mathbf{D})$. Since $mf' = fm'$ is total, by Lemma 1.5.2, f' is total. Then

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow{f} & D \end{array}$$

is a commutative diagram in $\mathsf{Total}(\mathbf{D})$. Now, we want to show that it is a pullback diagram. To do this, let $x : X \to C$ and $y : X \to B$ be maps in $\mathsf{Total}(\mathbf{D})$ such that $fx = my$. Since

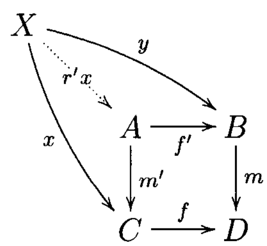
$$m'r'x = \overline{rf}x = x\overline{mrfx} = x\overline{rmy} = x\overline{y} = x,$$

and

$$f'r'x = rfm'r'x = rf\overline{rf}x = rfx = rmy = y,$$

and m' is monic, there is a unique map $r'x : X \to A$ in **Total**(**D**) such that

$$m'r'x = x \text{ and } f'r'x = y :$$



Hence (f', m') is a pullback of (f, m) in $\mathsf{Total}(\mathbf{D})$. □

Lemma 2.3.6 *Every $\mathcal{E}_{\mathbf{D}}$-map is pullback stable along an $\mathcal{M}_{\mathbf{D}}$-map in* $\mathsf{Total}(\mathbf{D})$.

PROOF: Let $e \in \mathcal{E}_{\mathbf{D}}$ and $m \in \mathcal{M}_{\mathbf{D}}$. Assume $rm = 1$ and $\overline{r} = mr$. Since $\overline{e} = 1$, by Lemma 2.3.5 the pullback of e along m exists. Write $e' = rem$ and $\overline{re} = m'r'$ with $r'm' = 1$. Then (e', m') is a pullback of (e, m) in $\mathsf{Total}(\mathbf{D})$. Compute

$$\begin{aligned}
\widehat{e'} &= \widehat{rem'} \\
&= \widehat{rem'\widehat{r'}} \text{ (since } r' \text{ is epic)} \\
&= \widehat{rem'r'} \text{ (by [}\mathbf{RR.4}\text{])} \\
&= \widehat{re\overline{re}} \\
&= \widehat{re} \\
&= \widehat{r\widehat{e}} \text{ (by [}\mathbf{RR.4}\text{])} \\
&= \widehat{r} \\
&= 1 \text{ (since } r \text{ is epic)}
\end{aligned}$$

and

$$\overline{e'} = \overline{rem'} = \overline{\overline{r}em'} = \overline{m'r'm'} = \overline{m'} = 1.$$

Then $e' \in \mathcal{E}_{\mathbf{D}}$, as desired. □

PROOF OF THEOREM 2.3.3. Combine Lemmas 2.3.4, 2.3.5, and 2.3.6. □

If $\mathbf{C}$ is a range restriction category, then by Proposition 2.3.2 $\mathsf{Split}_E(\mathbf{C})$ is a range restriction category with split restriction and so Theorem 2.3.3 is applicable to $\mathsf{Split}_E(\mathbf{C})$. Hence, we have:

Theorem 2.3.7 *If $\mathbf{C}$ is a range restriction category, then $\mathsf{Total}(\mathsf{Split}_E(\mathbf{C}))$ admits the $(\mathcal{E}, \mathcal{M})$-factorization system which is pullback stable along $\mathcal{M}$-maps and has pullbacks along $\mathcal{M}$-maps, where $\mathcal{E} = \{f : X \to Y \text{ in } \mathsf{Total}(\mathsf{Split}_E(\mathbf{C})) \mid \widehat{f} = 1_Y\}$ and $\mathcal{M} = \{m : X \to Y \text{ in } \mathsf{Total}(\mathsf{Split}_E(\mathbf{C})) \mid \exists\, r : Y \to X \text{ in } \mathsf{Split}_E(\mathbf{C}),\ rm = 1_X \text{ and } \overline{mr} = mr\}$.*

2.3.3 $\mathbf{rrCat}_s$ is 2-equivalent to $\mathcal{M}$StabFac

If $F : \mathbf{C} \to \mathbf{D}$ is a range restriction functor between two range restriction categories with split restriction, then we have a functor $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{C}) \to \mathsf{Total}(\mathbf{D})$ by restricting F to $\mathsf{Total}(\mathbf{C})$. The construction of pullbacks in $\mathsf{Total}(\mathbf{D})$ (see Lemma 2.3.5) yields that $\mathsf{Total}(F)$ preserves pullbacks along $\mathcal{M}_{\mathbf{C}}$-maps. Obviously, $\mathsf{Total}(F)\mathcal{E}_{\mathbf{C}} \subseteq \mathcal{E}_{\mathbf{D}}$, and $\mathsf{Total}(F)\mathcal{M}_{\mathbf{C}} \subseteq \mathcal{M}_{\mathbf{D}}$. Hence, we have a functor

$$\mathsf{Total}(F) : (\mathsf{Total}(\mathbf{C}), \mathcal{E}_{\mathbf{C}}, \mathcal{M}_{\mathbf{C}}) \to (\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$$

and therefore a functor $\mathsf{Total} : \mathbf{rrCat}_s \to \mathcal{M}\mathbf{StabFac}$ given by:

$$\begin{array}{ccc} \mathbf{C} & \mapsto & (\mathsf{Total}(\mathbf{C}), \mathcal{E}_{\mathbf{C}}, \mathcal{M}_{\mathbf{C}}) \\ {\scriptstyle F}\downarrow & \mapsto & \downarrow{\scriptstyle \mathsf{Total}(F)} \\ \mathbf{D} & \mapsto & (\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}}) \end{array}$$

If $F, G : \mathbf{C} \to \mathbf{D}$ are range restriction functors and $\alpha : F \to G$ is a natural transformation whose components are total, then we can form a natural transformation $\mathsf{Total}(\alpha) : \mathsf{Total}(F) \to \mathsf{Total}(G)$ by the components of α. For the naturality and the $\mathcal{M}_{\mathbf{D}}$-cartesianness of $\mathsf{Total}(\alpha)$, we need to check that for each map $m : A \to B$ in $\mathcal{M}_{\mathbf{C}}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(m)}\downarrow & & \downarrow{\scriptstyle G(m)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

is a pullback diagram in $\mathsf{Total}(\mathbf{D})$. To do this, we assume $rm = 1$ and $\overline{r} = mr$. Then

$$G(r) \cdot G(m) = 1 \text{ and } \overline{G(r)} = G(m) \cdot G(r),$$

$$F(r) \cdot F(m) = 1 \text{ and } \overline{F(r)} = F(m) \cdot F(r),$$

and

$$\overline{G(r) \cdot \alpha_B} = \overline{\alpha_A \cdot F(r)} = \overline{F(r)} = F(\overline{r}) = F(mr) = F(m) \cdot F(r).$$

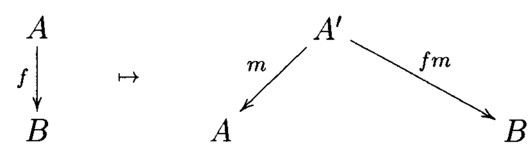
It follows that $F(m)$ splits the restriction idempotent $\overline{G(r) \cdot \alpha_B}$. Clearly,

$$G(r) \cdot \alpha_B \cdot F(m) = \alpha_A \cdot F(r) \cdot F(m) = \alpha_A.$$

So by Lemma 2.3.5 the last diagram is a pullback, as desired. These data now form a 2-functor $\mathsf{Total} : \mathbf{rrCat}_s \to \mathcal{M}\mathbf{StabFac}$.

Theorem 2.3.8 *The* 2*-functors* Total *and* Par *give an equivalence of* 2*-categories between* $\mathbf{rrCat}_s$ *and* $\mathcal{M}\mathbf{StabFac}$.

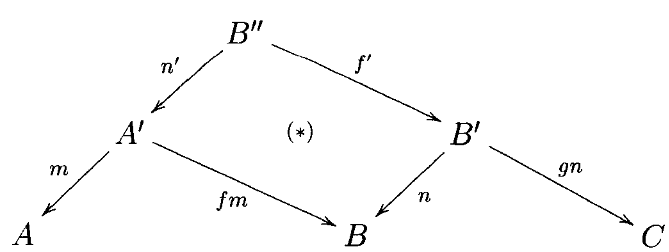
PROOF: In order to prove $\mathsf{Par} \circ \mathsf{Total} \cong 1_{\mathbf{rrCat}_s}$, for each range restriction category with split restriction structure $\mathbf{D}$, we define $\Phi_{\mathbf{D}} : \mathbf{D} \to \mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$ by



where m is determined by the conditions $\overline{f} = mr = \overline{r}$ and $rm = 1$. Since

$$\overline{fm} = \overline{\overline{f}m} = \overline{mrm} = \overline{m} = 1,$$

$\Phi_{\mathbf{D}}$ is well-defined. Clearly $\Phi_{\mathbf{D}}(1_A) = (1_A, 1_A)$ for each object A. For any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{D}$, assume that $\overline{f} = mr = \overline{r}, rm = 1$ and $\overline{g} = ns = \overline{s}, sn = 1$. Write $\overline{sfm} = n's' = \overline{s'}$ with $s'n' = 1$ and let $f' = sfmn'$. Then, by Lemma 2.3.5, (n', f') is a pullback of (n, fm) and so $\Phi_{\mathbf{D}}(g)\Phi_{\mathbf{D}}(f) = (mn', f'gn)$ since $(*)$ is a pullback:



But

$$mn's'r = \overline{msfmr} = \overline{mrnsfmr} = \overline{f}\,\overline{gf} = \overline{gf}$$

and $(s'r)(mn') = 1$, so mn' is the monic part of $\overline{gf}$. Notice that $gnf' = gnsfmn' = g\overline{g}fmn' = gfmn'$. Then $\Phi_{\mathbf{D}}(gf) = (mn', gfmn') = \Phi_{\mathbf{D}}(g)\Phi_{\mathbf{D}}(f)$. Hence $\Phi_{\mathbf{D}}$ is a functor. Since $\Phi_{\mathbf{D}}$ is the identity on objects, to prove $\mathsf{Par} \circ \mathsf{Total} \cong 1_{\mathbf{rrCat}_s}$, it suffices to show that $\Phi_{\mathbf{D}}$ is full and faithful. If (m, f) is a map in $\mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{M}_{\mathbf{D}})$, then there exists a unique map r such that $rm = 1$ and $mr = \overline{mr}$ and so $\Phi_{\mathbf{D}}(fr) = (m, frm) = (m, f)$ which means that $\Phi_{\mathbf{D}}$ is full. On the other hand, $\Phi_{\mathbf{D}}(g) = (m, f)$ yields $gm = f$ and $mr = \overline{g}$ so that $fr = gmr = g\overline{g} = g$. Faithfulness of $\Phi_{\mathbf{D}}$ follows, as desired.

For a $\mathcal{M}$-stable factorization system $(\mathbf{C}, \mathcal{E}, \mathcal{M})$, since the total maps in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ are the same as $\mathbf{C}$ and the monic parts of restriction idempotent in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ are just $\mathcal{M}$, we clearly have an isomorphism $\mathsf{Total} \circ \mathsf{Par} \cong 1_{\mathcal{M}\mathbf{StabFac}}$. Thus, Total and Par are part of an equivalence of 2-categories between $\mathbf{rrCat}_s$ and $\mathcal{M}\mathbf{StabFac}$. □

2.3.4 Example of a Restriction Category which is not a Range Restriction Category

Split range restriction categories rise essentially from $\mathcal{M}$-stable factorization systems while split restriction categories rise essentially from stable systems of monics. If $(\mathbf{C}, \mathcal{M})$ is an $\mathcal{M}$-category in which $\mathbf{C}$ does not admit any $\mathcal{M}$-stable $(\mathcal{E}, \mathcal{M})$-factorization system, then $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is not a range restriction category but a restriction category with the split restriction given by $\overline{(m, f)} = (m, m)$. Such an $\mathcal{M}$-category is as follows.

Let $\mathbf{Set}_{\text{ffib}}$ be the subcategory of $\mathbf{Set}$ with functions $f : A \to B$ such that $|f^{-1}(b)| < +\infty$ for each $b \in B$ as maps. Consider

$$\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}.$$

Then we have:

Proposition 2.3.9 $\mathbf{Set}_{\text{ffib}}$ *does not admit any* $(\mathcal{E}, \mathcal{M})$*-factorization system with*

$$\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}.$$

PROOF: We begin with:

Claim $\mathcal{M}$ *is a stable system of monics.*

Clearly, all isomorphisms are in $\mathcal{M}$. For any $i : A \to B, j : B \to C \in \mathcal{M}$, we have

$$|B \setminus i(A)| < +\infty \text{ and } |C \setminus j(B)| < +\infty.$$

Then

$$|C \setminus ji(A)| = |(C \setminus j(B)) \cup j(B \setminus i(A))| \leq |C \setminus j(B)| + |B \setminus i(A)| < +\infty,$$

and so $ji \in \mathcal{M}$. To prove the Claim, it suffices to prove that $\mathcal{M}$ is pullback stable.

For any $\mathcal{M}$-map $i : A \hookrightarrow B$ and any map $f : X \to B$ in $\mathbf{Set}_{\text{ffib}}$,

$$\begin{array}{ccc} f^{-1}(A) & \hookrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A & \hookrightarrow & B \end{array}$$

is a pullback. Since $f^{-1}(B \setminus A) = X \setminus f^{-1}(A)$, $|B \setminus A| < +\infty$, and since each $f^{-1}(b)$ is a finite set for each $b \in B$, we have

$$\begin{aligned} |X \setminus f^{-1}(A)| &= |f^{-1}(B \setminus A)| \\ &\leq \sum_{b \in B \setminus A} |f^{-1}(b)| \\ &< +\infty. \end{aligned}$$

Hence $f^{-1}(A) \hookrightarrow X$ is an $\mathcal{M}$-map, as desired.

Suppose, for contradiction, that $\mathbf{Set}_{\text{ffib}}$ admitted an $(\mathcal{E}, \mathcal{M})$-factorization system. Then $\mathcal{E} = \mathcal{M}^{\perp}$. Clearly, $0 : \{*\} \to \mathbb{N}$ is a map of $\mathbf{Set}_{\text{ffib}}$ but 0 is not an $\mathcal{M}$-map. Suppose that $0 = m_0 e_0$, where $e_0 : \{*\} \to X$ is an $\mathcal{E}$-map and $m_0 : X \to \mathbb{N}$ is $\mathcal{M}$-map. Then $e_0 \perp \mathcal{M}$ and so e_0 is surjective. In fact, if e_0 were not surjective, then there existed $x_0 \in X \setminus e_0(\{*\})$. Note that

$$\begin{array}{ccc} \{*\} & \xrightarrow{e_0} & X \\ {\scriptstyle e_0}\downarrow & & \| \\ X \setminus \{x_0\} & \overset{i}{\hookrightarrow} & X \end{array}$$

is commutative in $\mathbf{Set}_{\text{ffib}}$ and $i \in \mathcal{M}$. But, clearly, in the last diagram there does not exist the diagonal map $x : X \to X \setminus \{x_0\}$ such that $xe_0 = e_0$ and $ix = 1$ since

$ix(x_0) \neq x_0$, which contradicts to $e_0 \perp \mathcal{M}$.

Since e_0 is a surjection, $X = \{e_0(*)\}$. Hence

$$|\mathbb{N} \setminus m_0(X)| = +\infty,$$

which contradicts to that m_0 is an $\mathcal{M}$-map. Hence $\mathbf{Set}_{\mathrm{ffib}}$ does not have $(\mathcal{E}, \mathcal{M})$-factorization system. $\square$

Chapter 3

Restriction Categories and Fibrations

For a given restriction category $\mathbf{C}$, Cockett and Lack [7] constructed a functor $\mathrm{RId}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Poset}$ in which each poset $\mathrm{RId}_{\mathbf{C}}(C)$ is a meet semilattice. Hence, by Grothendieck construction (See Section 3.1 below), each restriction category gives rise to a fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ in which each fiber is a meet semilattice. In this chapter, we study how restriction categories connect with fibrations. We shall first recall the notions of fibrations, indexed categories, and their relation by Grothendieck construction. Secondly, we introduce the notion of stable meet semilattice fibrations and show that such fibrations produce restrictions and provide a bridge between the category of restriction categories and the category of categories. Thirdly, we produce Cockett-Lack's free restriction category structures over categories using the special stable meet semilattice fibrations $\Delta_{\mathbf{C}}$ for each category $\mathbf{C}$. Finally, we introduce the notion of restriction fibrations and show that restriction fibrations are the same as restriction categories.

3.1 Preliminaries for Fibrations

This section is devoted to the presentation of the fundamentals of the theory of fibrations.

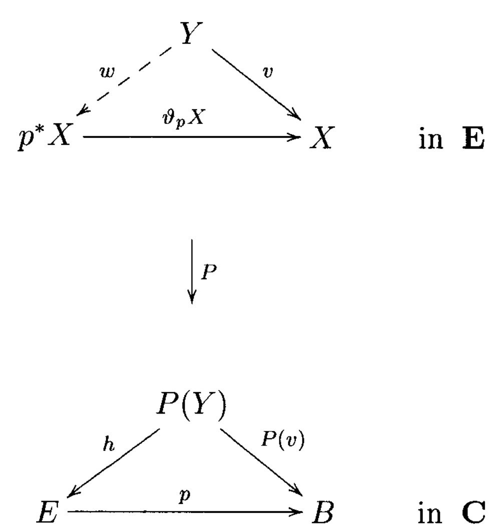
3.1.1 Definition of Fibrations

Definition 3.1.1 *Let* $P : \mathbf{E} \to \mathbf{C}$ *be a functor and* $p : E \to B$ *a map of* $\mathbf{C}$. The fiber *of* P *at* B *is the non-full subcategory* $\mathbf{E}(B)$ *of* $\mathbf{E}$ *whose objects are in* $P^{-1}(B)$ *(i.e., those objects* A *of* $\mathbf{E}$ *with* $P(A) = B$*) and whose maps* $f : A \to A'$ *are* $\mathbf{E}$*-maps such that* $P(f) = 1_B$. *If* $X \in \mathbf{E}(B)$*, then a map* $\vartheta_p X : p^*X \to X$ *of* $\mathbf{C}$ *is* a cartesian lifting over p at X *if*

[F.1] $P(\vartheta_p X) = p$;

[F.2] *For any map* $v : Y \to X$ *of* $\mathbf{E}$ *and any map* $h : P(Y) \to E$ *in* $\mathbf{C}$ *satisfying* $ph = P(v)$*, there is a unique* $w : Y \to p^*X$ *in* $\mathbf{E}$ *such that*

$$\vartheta_p X \cdot w = v \text{ and } P(w) = h.$$



A functor $P : \mathbf{E} \to \mathbf{C}$ *is called* a fibration *if for any map* $p : E \to B$ *in* $\mathbf{C}$ *and every object* X *in* $\mathbf{E}(B)$ *there is a cartesian lifting* $(p^*X,\ \vartheta_p X)$ *over* p *at* X. *A functor* $P : \mathbf{E} \to \mathbf{C}$ *is called* an opfibration *if* P^{op} *is a fibration.* P *is a* bifibration *if both* P

and P^{op} are fibrations.

3.1.2 Examples of Fibrations

1. *Identity Functors.* For every category $\mathbf{C}$, the identity functor $1_{\mathbf{C}}$ is a bifibration.

2. *Basic Fibrations.* If $\mathbf{C}$ is a category with pullbacks, and if $\mathbf{C}^2$ is the map category of $\mathbf{C}$ with

   objects: maps $f : E \to B$ in $\mathbf{C}$,

   maps: maps from $f : E \to B$ to $f' : E' \to B'$ are pairs (u, v) of maps in $\mathbf{C}$ such that $f'u = vf$, where $u : E \to E'$ and $v : B \to B'$ are $\mathbf{C}$-maps:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{v} & B' \end{array}$$

   composition and **identities**: defined as follows:

$$(u', v')(u, v) = (u'u, v'v)$$

   and

$$1_{f:E\to B} = (1_E, 1_B),$$

   then the codomain functor

$$\partial :\ \mathbf{C}^2 \to \mathbf{C} :\ (f : A \to B) \mapsto B,\ (u,\ v) \mapsto v$$

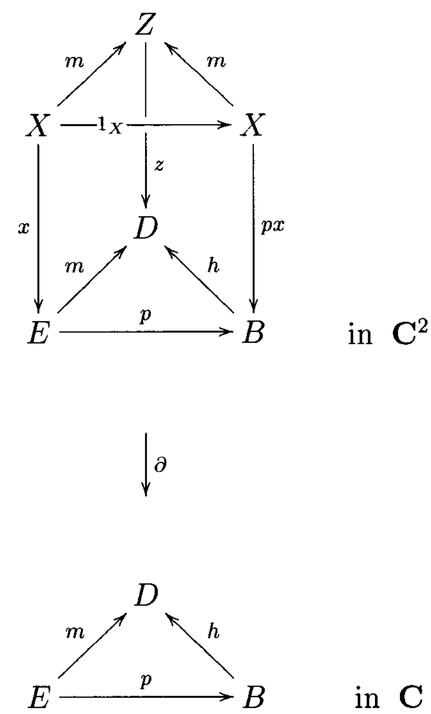
is a fibration, called *the basic fibration of* $\mathbf{C}$. In fact, for any map $p : E \to B$ in $\mathbf{C}$ and an object $(x : X \to B)$ in $\partial^{-1}(B)$, the following pullback diagram:

$$\begin{array}{ccc} p^*X & \xrightarrow{\vartheta_p X} & E' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle x} \\ B & \xrightarrow{p} & B' \end{array}$$

yields a map $(\vartheta_p X,\ p) : (f : \partial^* X \to E) \to (x : X \to B)$ in $\mathbf{C}^2$, which is a cartesian lifting over $(p : E \to B)$ at $(x : X \to B)$.

On the other hand, ∂ is an *opfibration* (i.e., ∂^{op} is a fibration). In fact, for any map $p : E \to B$ in $\mathbf{C}$ and any object $(x : X \to E)$ in $\partial^{-1}(E)$, an *opcartesian lifting* over $(p : E \to B)$ at $(x : X \to E)$ is given by

$$(1_X,\ p) : (x : X \to E) \to (px : X \to B) :$$



Hence ∂ *is a bifibration.*

3. *Modules.* Let MOD be the category defined as follows:

- An object of MOD is a pair (R, M), where $R \in \mathbf{CRng}_1$, $M \in \mathbf{Mod}\text{-}R$;
- A map $(f, u) : (R, M) \to (R', M')$ has $f : R' \to R$ a map (unital commutative ring homomorphism) of $\mathbf{CRng}_1$ and $u : M \to M' \otimes_{R'} R$ a map of $\mathbf{Mod}\text{-}R$;
- If $(f, u) : (R, M) \to (R', M')$, $(g, v) : (R', M') \to (R'', M'')$ are maps of MOD, then

$$(g, v)(f, u) : (R, M) \to (R'', M'')$$

is given by

$$(g, v)(f, u) = (fg,\ (v \otimes_{R'} 1_R)u).$$

Then the functor mod: MOD $\to (\mathbf{CRng}_1)^{\text{op}}$ given by

$$\begin{array}{ccc} (R, M) & \mapsto & R \\ {\scriptstyle (f,u)}\downarrow & \mapsto & \downarrow{\scriptstyle f} \\ (R', M') & \mapsto & R' \end{array}$$

is a fibration. In fact, for any unital commutative ring homomorphism $f : R \to S$ and any $(R, M) \in \text{mod}^{-1}(R)$, $(f, 1_{M \otimes_R S}) : (S, M \otimes_R S) \to (R, M)$ is a cartesian lifting over f at (R, M).

4. *Topologies.* Let $F : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. Then F is a fibration: for any map $p : E \to B$ in **Set** and for $B \in \mathbf{Top}$, if E is equipped with the coarsest topology which makes $p : E \to B$ continuous, then $p : E \to B$ is a cartesian lifting over p at B. F is also an opfibration: for any map $p : E \to B$ in **Set** and for $E \in \mathbf{Top}$, $p : E \to B$ is a cocartesian lifting over p at E if B is equipped with the finest topology which makes $p : E \to B$ continuous. Hence F is *a bifibration.*

3.1.3 Indexed Categories vs Fibrations

We shall see that fibrations and indexed categories are essentially the same.

Indexed Categories

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-*indexed category* $\mathbb{A}$ is a pseudo-functor $\mathbb{A} : \mathbf{C}^{\text{op}} \to \mathbf{CAT}$ such that for every $f : E \to D$, $g : D \to C$ in $\mathbf{C}$, there are natural isomorphisms

$$i^D : 1_{\mathbb{A}^D} \to (1_D)^*, \quad j^{f,g} : f^* g^* \to (gf)^*$$

which satisfy that

$$\begin{array}{ccc} f^* & \xrightarrow{f^* i^D} & f^*(1_D)^* \\ {\scriptstyle i^E f^*}\downarrow & \searrow^{1_{f^*}} & \downarrow{\scriptstyle j^{f,1_D}} \\ (1_E)^* f^* & \xrightarrow[j^{1_E,f}]{} & f^* \end{array}$$

and

$$\begin{array}{ccc} f^*g^*h^* & \xrightarrow{f^* j^{g,h}} & f^*(hg)^* \\ {\scriptstyle j^{f,g}h^*}\downarrow & & \downarrow{\scriptstyle j^{f,hg}} \\ (gf)^*h^* & \xrightarrow[j^{gf,h}]{} & (hgf)^* \end{array}$$

commute, where $\mathbb{A}^D = \mathbb{A}(D)$ and x^* denotes $\mathbb{A}(x) : \mathbb{A}^D \to \mathbb{A}^E$ for any map $x : E \to D$ in $\mathbf{C}$.

For example,

$$\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

given by $B \longmapsto \mathbf{C}/B$ and $(f : E \to B) \longmapsto f^* : \mathbf{C}/B \to \mathbf{C}/E$, the pullback functor along f, is a $\mathbf{C}$-indexed category, and we call it the *basic* $\mathbf{C}$*-indexed category.*

Indexed Categories Give Rise to Fibrations by Grothendieck Construction

Given a $\mathbf{C}$-indexed category $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, one defines a category $G(\mathbf{C},\ \mathbb{A})$ as follows:

- An object of $G(\mathbf{C}, \mathbb{A})$ is a pair (C, x), where C is an object of $\mathbf{C}$ and x is an object of $\mathbb{A}(C)$;

- A map $(f,u):(C,x)\to(C',x')$ consists of a $\mathbf{C}$-map $f:C\to C'$ and a $\mathbb{A}(C)$-map $u:x\to\mathbb{A}(f)(x')$;
- If $(f,u):(C,x)\to(C',x'),(g,v):(C',x')\to(C'',x'')$ are maps of $G(\mathbf{C},\mathbb{A})$, then

  $$(g,v)(f,u):(C,x)\to(C'',x'')$$

  is given by

  $$(g,v)(f,u)=(gf,\mathbb{A}(f)(v)u).$$

The projection functor $P:G(\mathbf{C},\mathbb{A})\to\mathbf{C}$ given by

$$\begin{array}{ccc} (C,x) & \mapsto & C \\ {\scriptstyle (f,u)}\downarrow & \mapsto & \downarrow{\scriptstyle f} \\ (C',x') & \mapsto & C' \end{array}$$

is a fibration. This process is called *the Grothendieck construction* (see [1]). So an indexed category gives rise to a fibration by *the Grothendieck construction.*

Clearly, mod: MOD $\to(\mathbf{CRng}_1)^{\mathrm{op}}$ is given by the Grothendieck construction applied to the $(\mathbf{CRng}_1)^{\mathrm{op}}$-indexed category $R\mapsto$**Mod**-R, in Example 3.1.2 (3). Hence it is a fibration.

Fibrations Gives Rise to Indexed Categories

If $P:\mathbf{E}\to\mathbf{C}$ is a fibration and $p:E\to B$ is a map in $\mathbf{C}$, then we have *the inverse-image functor*

$$p^*:\mathbf{E}(B)\to\mathbf{E}(E)$$

given by $A \mapsto p^*A$ and obvious assignments on maps, and *a cleavage*

$$\vartheta_p : J_E p^* \to J_B,$$

where $J_B : \mathbf{E}(B) \to \mathbf{E}$ is the inclusion functor and $P\vartheta_p = \triangle p : \triangle E \to \triangle B$ is the constant natural transformation. Hence one gets a *pseudo-functor*

$$(\ )^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

given by

$$\begin{array}{ccc} B & \mapsto & \mathbf{E}(B) \\ {\scriptstyle p}\downarrow & \mapsto & \downarrow{\scriptstyle p^*} \\ E & \mapsto & \mathbf{E}(E) \end{array}$$

since there are the uniquely determined natural equivalences

$$i_B : 1_{\mathbf{E}(B)} \to (1_B)^* \ \text{ and } \ j_{p,q} : q^*p^* \to (pq)^*$$

such that

$$\vartheta_{1_B} \cdot J_B i_B = 1_{J_B},\ PJ_B i_B = \triangle 1_B,$$

and

$$\vartheta_{pq} \cdot J_X j_{p,q} = \vartheta_p \cdot \vartheta_q p^*,\ PJ_X j_{p,q} = \triangle 1_X,$$

for any $p : E \to B$ and $q : X \to E$ in $\mathbf{C}$ by the definition of the cartesian lifting. This means that $(\ )^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ is a $\mathbf{C}$-indexed category.

A fibration gives rise to an indexed category by the above process. For example, the basic fibration yields the basic **C**-indexed category given by the sliced categories and pullback functors.

On the other hand, given a **C**-indexed category $\mathbb{A} : \mathbf{C}^{\text{op}} \to \mathbf{CAT}$, as we already saw, by the Grothendieck construction, one may construct a fibration. Any **C**-indexed category essentially rises in this way (see [16]).

3.2 Stable Meet Semilattice Fibrations and Restriction Categories

In this section, we shall introduce the notion of stable meet semilattice fibrations and show that such fibrations produce restriction categories and provide a bridge between the category of restriction categories and the category of categories.

3.2.1 Stable Meet Semilattice Fibrations

Definition 3.2.1 A stable meet semilattice fibration *is a fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *in which for each map* $f : X \to Y$*, the inverse image functor* $f^* : \delta_{\mathbf{X}}^{-1}(Y) \to \delta_{\mathbf{X}}^{-1}(X)$ *is a stable meet semilattice homomorphism.*

Clearly, stable meet semilattice fibrations are precisely those fibrations given by indexed categories $\mathbf{X}^{\text{op}} \to \mathbf{msLat}$.

For example, for any category **C**, the identity fibration $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is a stable meet semilattice fibration.

Restriction Categories Give Rise to Stable Meet Semilattice Fibrations

Each restriction category gives rise to a stable meet semilattice fibration as shown in Lemma 3.2.3 below.

Suppose that $\mathbf{C}$ is a restriction category. As in [7], one can form the category $\mathbf{r}(\mathbf{C})$ with the following data:

objects: (X, e_X), where X is an object of $\mathbf{C}$ and each e_X is a restriction idempotent on X;

maps: a map from (X, e_X) to (Y, e_Y) is a map $f : X \to Y$ in $\mathbf{C}$ such that $e_X = \overline{e_Y f e_X}$;

composition and **identities** are formed as in $\mathbf{C}$.

If $f : (X, e_X) \to (Y, e_Y)$ is a map of $\mathbf{r}(\mathbf{C})$, then $e_X = \overline{e_Y f e_X}$ and so $e_X = \overline{f}\,\overline{e_Y f e_X} = \overline{f} e_X$. It follows that $e_X = e_X \overline{f} e_X = \overline{e_X \overline{f}} e_X$. Hence $\overline{f} : (X, e_X) \to (X, e_X)$ is also a map of $\mathbf{r}(\mathbf{C})$ and therefore $\mathbf{r}(\mathbf{C})$ is a restriction category with the restriction of a map $f : (X, e_X) \to (Y, e_Y)$ given by $\overline{f} : (X, e_X) \to (X, e_X)$, where $\overline{f}$ is the restriction of f in category $\mathbf{C}$. Moreover, we have the forgetful functor $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ which forgets the restriction idempotents. Since $\partial_{\mathbf{C}}(\overline{f}) = \overline{f} = \overline{\partial_{\mathbf{C}}(f)}$ for any map f in $\mathbf{r}(\mathbf{C})$, $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a restriction functor.

Moreover, if $\mathbf{C}$ is a range restriction category, then, of course, it is a restriction category and so we have the restriction category $\mathbf{r}(\mathbf{C})$. Now we form the subcategory $\mathbf{rr}(\mathbf{C})$ of $\mathbf{r}(\mathbf{C})$ with the same objects as $\mathbf{r}(\mathbf{C})$ but with maps $f : (X, e_X) \to (Y, e_Y)$ in $\mathbf{r}(\mathbf{C})$ such that $e_Y = \widehat{f} e_Y$. If $f : (X, e_X) \to (Y, e_Y)$ is a map in $\mathbf{rr}(\mathbf{C})$, then $e_Y = \widehat{f} e_Y = \overline{\widehat{f}} e_Y = \overline{e_Y \widehat{f}} e_Y$ and so both $\overline{f} : (X, e_X) \to (X, e_X)$ and $\widehat{f} : (Y, e_Y) \to (Y, e_Y)$ are maps in $\mathbf{rr}(\mathbf{C})$. Hence $\mathbf{rr}(\mathbf{C})$ is a range restriction category with the range and restriction structures given by those structures in $\mathbf{C}$. So we have:

Lemma 3.2.2 *If* $\mathbf{C}$ *is a restriction category, then so is* $\mathbf{r}(\mathbf{C})$. *If* $\mathbf{C}$ *is a range restriction category, then so is* $\mathbf{rr}(\mathbf{C})$.

Lemma 3.2.3 *If* $\mathbf{C}$ *is a restriction category, then the forgetful functor* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *is a stable meet semilattice fibration.*

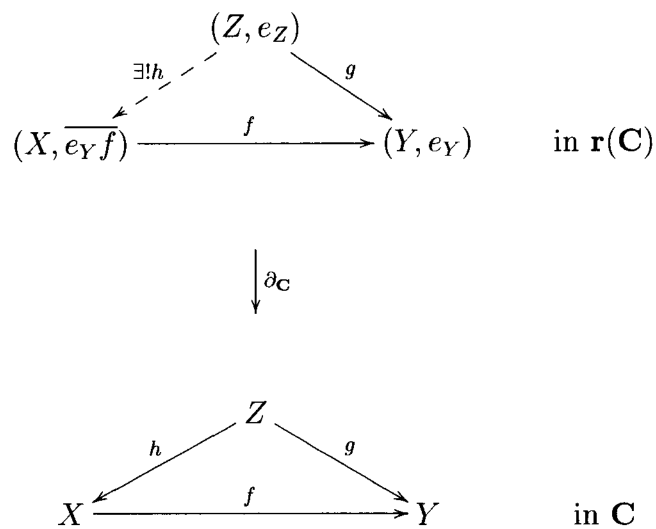
PROOF: For any map $f : X \to Y$ in $\mathbf{C}$ and any object $(Y, e_Y) \in \partial_{\mathbf{C}}^{-1}(Y)$, $f : (X, \overline{e_Y f}) \to (Y, e_Y)$ is a map of $\mathbf{r}(\mathbf{C})$ since $\overline{e_Y f}^2 = \overline{e_Y f}$. Moreover, $f : (X, \overline{e_Y f}) \to (Y, e_Y)$ is the cartesian lifting of a map $f : X \to Y$ at (Y, e_Y). In fact, for any map $g : (Z, e_Z) \to (Y, e_Y)$ in $\mathbf{r}(\mathbf{C})$, we have

$$e_Z = \overline{e_Y g} e_Z.$$

If $h : Z \to X$ is a map such that $fh = g$ in $\mathbf{C}$, then

$$e_Z = \overline{e_Y f h} e_Z = \overline{\overline{e_Y f} h} e_Z.$$

Hence $h : (Z, e_Z) \to (X, \overline{e_Y f})$ is a map such that $\partial_{\mathbf{C}}(h) = h$ and $fh = g$ in $\mathbf{r}(\mathbf{C})$:



The uniqueness of the map $h : (Z, e_Z) \to (X, \overline{e_Y f})$ in $\mathbf{r}(\mathbf{C})$ such that $fh = g$ in $\mathbf{r}(\mathbf{C})$ is obvious. Hence $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a fibration. Note that each fiber

$$\partial_{\mathbf{C}}^{-1}(X) = \{(X, e_X) \,|\, e_X : X \to X \text{ is a restriction idempotent on } X\}$$

is a meet semilattice with the order given by

$$(X, e_X) \leq (X, e'_X) \Leftrightarrow e_X = e'_X e_X$$

which is equivalent to saying that there is a map from (X, e_X) to (X, e'_X) in $\partial_{\mathbf{C}}^{-1}(X)$, with the binary meet given by

$$(X, e_X) \wedge (X, e'_X) = (X, e_X e'_X),$$

and with $(X, 1_X)$ as the top element. Obviously, for any map $f : X \to Y$, $f^* : \partial_{\mathbf{C}}^{-1}(Y) \to \partial_{\mathbf{C}}^{-1}(X)$, sending (Y, e_Y) to $(X, \overline{e_Y f})$, is a stable meet semilattice homomorphism. Hence $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration. □

3.2.2 The Construction $\mathcal{S}_s$

Suppose that $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a stable meet semilattice fibration. Then we can form $\mathcal{S}_s(\delta_{\mathbf{X}})$ with the following data:

objects: $A \in \mathrm{ob}\mathbf{X}$;

maps: $(f, \sigma) : A \to B$, where $f : A \to B$ is a map in $\mathbf{X}$ and $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$ is such that $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$;

composition: For any map $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : B \to C$,

$$(g, \sigma_2)(f, \sigma_1) = (gf, \sigma_1 \wedge f^*(\sigma_2));$$

identities: $1_A = (1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)})$.

In order to prove that $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a category, we must first check that the composition and identities are well-defined. For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : B \to C$, we have

$$\sigma_1 \in \delta_{\mathbf{X}}^{-1}(A), \sigma_1 \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}),$$

and

$$\sigma_2 \in \delta_{\mathbf{X}}^{-1}(B), \sigma_2 \leq g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}).$$

Then

$$f^*(\sigma_2) \wedge \sigma_1 \leq f^*(\sigma_2) \leq f^*(g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) = (gf)^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}),$$

and

$$f^*(\sigma_2) \wedge \sigma_1 \in f^*(\delta_{\mathbf{X}}^{-1}(B)) \wedge \{\sigma_1\} \in \delta_{\mathbf{X}}^{-1}(A),$$

and so the composition is well-defined. Since

$$1_A^*(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = 1_{\delta_{\mathbf{X}}^{-1}(A)}(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{X}}^{-1}(A)},$$

clearly the identities is also well-defined. We must also check the identity and associative laws. For any maps $(f, \sigma_1) : A \to B$, $(g, \sigma_2) : B \to C$ and $(h, \sigma_3) : C \to D$,

$$\begin{aligned}
(h, \sigma_3)((g, \sigma_2)(f, \sigma_1)) &= (h, \sigma_3)(gf, \sigma_1 \wedge f^*(\sigma_2)) \\
&= (hgf, \sigma_1 \wedge f^*(\sigma_2) \wedge (gf)^*(\sigma_3)) \\
&= (hgf, \sigma_1 \wedge f^*(\sigma_2) \wedge f^*g^*(\sigma_3)) \\
&= (hgf, \sigma_1 \wedge f^*(\sigma_2 \wedge g^*(\sigma_3))) \\
&= (hg, \sigma_2 \wedge g^*(\sigma_3))(f, \sigma_1) \\
&= ((h, \sigma_3)(g, \sigma_2))(f, \sigma_1),
\end{aligned}$$

$$(f, \sigma_1)(1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) = (f, \top_{\delta_{\mathbf{X}}^{-1}(A)} \wedge 1_A^*(\sigma_1)) = (f, \sigma_1 \wedge \top_{\delta_{\mathbf{X}}^{-1}(A)}) = (f, \sigma_1),$$

and

$$(1_B, \top_{\delta_{\mathbf{X}}^{-1}(B)})(f, \sigma_1) = (f, \sigma_1 \wedge f^*(\top_{\delta_{\mathbf{X}}^{-1}(A)})) = (f, \sigma_1),$$

as desired. Hence $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a category. Furthermore, $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a restriction category with the restriction given by $\overline{(f,\sigma)} = (1_A, \sigma)$ for any map $(f,\sigma) : A \to B$ as shown by the following proposition.

Proposition 3.2.4 $\mathcal{S}_s(\delta_{\mathbf{X}})$ *is a restriction category with the restriction given by* $\overline{(f,\sigma)} = (1_A, \sigma)$ *for any map* $(f,\sigma) : A \to B$.

PROOF: We already proved that $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a category. Clearly, $(1_A, \sigma) : A \to A$ is a map in $\mathcal{S}_s(\delta_{\mathbf{X}})$. So it suffices to check that $\overline{(f,\sigma)} = (1_A, \sigma)$ satisfies the four restriction axioms.

[R.1] For any map $(f,\sigma) : A \to B$,

$$(f,\sigma)\overline{(f,\sigma)} = (f,\sigma)(1_A,\sigma) = (f, \sigma \wedge 1_A^*(\sigma)) = (f, \sigma \wedge 1_{\delta_{\mathbf{X}}^{-1}(A)}(\sigma)) = (f,\sigma).$$

[R.2] For any maps $(f,\sigma_1) : A \to B$ and $(g,\sigma_2) : A \to C$,

$$\begin{aligned} \overline{(f,\sigma_1)}\ \overline{(g,\sigma_2)} &= (1_A,\sigma_1)(1_A,\sigma_2) \\ &= (1_A, \sigma_1 \wedge \sigma_2) \\ &= (1_A, \sigma_2 \wedge \sigma_1) \\ &= (1_A,\sigma_2)(1_A,\sigma_1) \\ &= \overline{(g,\sigma_2)}\ \overline{(f,\sigma_1)}. \end{aligned}$$

[R.3] For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : A \to C$,

$$\begin{aligned}
\overline{(g, \sigma_2)\overline{(f, \sigma_1)}} &= \overline{(g, \sigma_2)(1_A, \sigma_1)} \\
&= \overline{(g, \sigma_1 \wedge 1_A^*(\sigma_2))} \\
&= \overline{(g, \sigma_1 \wedge \sigma_2)} \\
&= (1_A, \sigma_1 \wedge \sigma_2) \\
&= (1_A, \sigma_2)(1_A, \sigma_1) \\
&= \overline{(g, \sigma_2)}\ \overline{(f, \sigma_1)}.
\end{aligned}$$

[R.4] For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : B \to C$, we have

$$\begin{aligned}
\overline{(g, \sigma_2)}(f, \sigma_1) &= (1_B, \sigma_2)(f, \sigma_1) \\
&= (f, \sigma_1 \wedge f^*(\sigma_2)),
\end{aligned}$$

and

$$\begin{aligned}
(f, \sigma_1)\overline{(g, \sigma_2)(f, \sigma_1)} &= (f, \sigma_1)\overline{(gf, \sigma_1 \wedge f^*(\sigma_2))} \\
&= (f, \sigma_1)(1_A, \sigma_1 \wedge f^*(\sigma_2)) \\
&= (f, \sigma_1 \wedge f^*(\sigma_2) \wedge 1_A^*(\sigma_1)) \\
&= (f, \sigma_1 \wedge f^*(\sigma_2)).
\end{aligned}$$

Hence $\overline{(g, \sigma_2)}(f, \sigma_1) = (f, \sigma_1)\overline{(g, \sigma_2)(f, \sigma_1)}$.

□

Examples

1. Suppose that $\mathbf{C}$ is a category. Then $\mathcal{S}_s(1_{\mathbf{C}}) = \mathbf{C}$, which is a restriction category with the trivial restriction structure.

2. For each restriction category $\mathbf{C}$, $\mathcal{S}_s(\partial_{\mathbf{C}})$ is the restriction category with the same objects as $\mathbf{C}$ while a map from A to B in $\mathcal{S}_s(\partial_{\mathbf{C}})$ is a pair (f, e) with a map $f : A \to B$ in $\mathbf{C}$ and a restriction idempotent $e \leq \overline{f}$ over A in $\mathbf{C}$, the composition is given by $(g, e_B)(f, e_A) = (gf, e_A \wedge \overline{e_B f}) = (gf, \overline{e_B f} e_A)$ for any maps $(f, e_A) : A \to B$ and $(g, e_B) : B \to C$, and the restriction is given by $\overline{(f, e_A)} = (1_A, e_A)$.

3.2.3 Category of Stable Meet Semilattice Fibrations and $\mathbf{rCat}_0$

First, we form the category of stable meet semilattice fibrations.

Category of Stable Meet Semilattice Fibrations

Let $\mathbf{sFib}_0$ be the category with

objects: stable meet semilattice fibrations: $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$;

maps: a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is a pair (F, F'), where $F : \mathbf{X} \to \mathbf{Y}$ and $F' : \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}}$ are functors such that

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \xrightarrow{F'} & \tilde{\mathbf{Y}} \\ {\scriptstyle \delta_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{Y}}} \\ \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \end{array}$$

commutes and for any map $f : A \to B$ in $\mathbf{X}$ and any $\sigma, \sigma' \in \delta_{\mathbf{X}}^{-1}(B)$, the following conditions are satisfied:

[sfM.1] $F'(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{Y}}^{-1}(F(A))}$,

[sfM.2] $F'(\sigma \wedge \sigma') = F'(\sigma) \wedge F'(\sigma')$,

[sfM.3] $F'(f^*(\sigma)) = (F(f))^*(F'(\sigma))$.

That is, a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is a pair (F, α), where $F : \mathbf{X} \to \mathbf{Y}$ is a functor and α is a natural transformation:

$$\begin{array}{ccc} \mathbf{X}^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & \mathbf{Y}^{\mathrm{op}} \\ & {\scriptstyle ()^*_{\delta_{\mathbf{X}}}} \searrow \quad \Uparrow\alpha \quad \swarrow {\scriptstyle ()^*_{\delta_{\mathbf{Y}}}} & \\ & \mathbf{sLat} & \end{array}$$

composition: for any maps $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ and $(G, G') : (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y}) \to (\delta_{\mathbf{Z}} : \tilde{\mathbf{Z}} \to \mathbf{Z})$, $(G, G')(F, F') = (GF, G'F')$;

identities: $1_{(\delta_{\mathbf{X}}:\tilde{\mathbf{X}}\to\mathbf{X})} = (1_{\mathbf{X}}, 1_{\tilde{\mathbf{X}}})$.

Functor $\mathcal{S}_s : \mathbf{sFib}_0 \to \mathbf{rCat}_0$

Recall that each stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ gives rise to a restriction category $\mathcal{S}_s(\delta_{\mathbf{X}})$. Suppose now that $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is map in $\mathbf{sFib}_0$. Then we define $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$ by

$$\begin{array}{ccc} A & \mapsto & F(A) \\ {\scriptstyle (f,\sigma)}\downarrow & \mapsto & \downarrow{\scriptstyle (F(f),F'(\sigma))} \\ B & \mapsto & F(B) \end{array}$$

If $(f, \sigma) : A \to B$ is a map in $\mathcal{S}_s(\delta_{\mathbf{X}})$, then

$$\sigma \in \delta_{\mathbf{X}}^{-1}(A) \text{ and } \sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$$

and so

$$(\delta_{\mathbf{Y}} F')(\sigma) = (F\delta_{\mathbf{X}})(\sigma) = F(A)$$

and

$$\begin{aligned} F'(\sigma) &\leq F'(f^*(\top_{\delta_{\mathbf{X}}^{-1}(A)})) \text{ (by [}\mathbf{sfM.2}\text{])} \\ &= (F(f))^*(F'(\top_{\delta_{\mathbf{X}}^{-1}(A)})) \text{ (by [}\mathbf{sfM.3}\text{])} \\ &= (F(f))^*(\top_{\delta_{\mathbf{Y}}^{-1}(F(A))}) \text{ (by [}\mathbf{sfM.1}\text{]).} \end{aligned}$$

Hence $(F(f), F'(\sigma)) : F(A) \to F(B)$ is a map in $\mathcal{S}_s(\delta_{\mathbf{Y}})$ and therefore $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$ is well-defined. Now we have:

Lemma 3.2.5 *If $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is map in* $\mathbf{sFib}_0$, *then $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$, given by taking $(f, \sigma) : A \to B$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$ to $(F(f), F'(\sigma)) : F(A) \to F(B)$ in $\mathcal{S}_s(\delta_{\mathbf{Y}})$, is a restriction functor.*

PROOF: We already knew that $\mathcal{S}_s(F, F')$ is well-defined. Clearly,

$$\begin{aligned} \mathcal{S}_s(F, F')(1_A) &= \mathcal{S}_s(F, F')(1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) \\ &= (F(1_A), F'(\top_{\delta_{\mathbf{X}}^{-1}(A)})) \\ &= (1_{F(A)}, \top_{\delta_{\mathbf{Y}}^{-1}(F(A))}) \text{ (by [}\mathbf{sfM.1}\text{])} \\ &= 1_{F(A)} \\ &= 1_{\mathcal{S}_s(F,F')(A)}. \end{aligned}$$

For any map $(f, \sigma) : A \to B$ and $(g, \sigma') : B \to C$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$,

$$\begin{aligned}
\mathcal{S}_s(F, F')((g, \sigma')(f, \sigma)) &= \mathcal{S}_s(F, F')(gf, \sigma \wedge f^*(\sigma')) \\
&= (F(gf), F'(\sigma \wedge f^*(\sigma'))) \\
&= (F(g) \cdot F(f), F'(\sigma) \wedge F'(f^*(\sigma'))) \text{ (by [}\mathbf{sfM.2}\text{])} \\
&= (F(g) \cdot F(f), F'(\sigma) \wedge (F(f))^*(F'(\sigma))) \text{ (by [}\mathbf{sfM.3}\text{])} \\
&= (F(g), F'(\sigma))(F(f), F'(\sigma)) \\
&= \mathbf{r}(F, F')(g, \sigma') \cdot \mathbf{r}(F, F')(f, \sigma).
\end{aligned}$$

Hence $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$ is a functor.

Note that, for any map $(f, \sigma) : A \to B$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$,

$$\begin{aligned}
\mathcal{S}_s(F, F')(\overline{(f, \sigma)}) &= \mathcal{S}_s(F, F')((1_A, \sigma)) \\
&= (1_{F(A)}, F'(\sigma)) \\
&= \overline{(F(f), F'(\sigma))} \\
&= \overline{\mathcal{S}_s(F, F')(f, \sigma)}.
\end{aligned}$$

Hence $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$ is a restriction functor. □

Furthermore, it is easy to check that $\mathcal{S}_s : \mathbf{sFib}_0 \to \mathbf{rCat}_0$ given by

$$\begin{array}{ccc}
(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) & \mapsto & \mathcal{S}_s(\delta_{\mathbf{X}}) \\
{\scriptstyle (F,F')}\downarrow & \mapsto & \downarrow{\scriptstyle \mathcal{S}_s(F,F')} \\
(\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y}) & \mapsto & \mathcal{S}_s(\delta_{\mathbf{Y}})
\end{array}$$

is a functor. So we have:

Lemma 3.2.6 $\mathcal{S}_s : \mathbf{sFib}_0 \to \mathbf{rCat}_0$, *given by sending* $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ *in* $\mathbf{sFib}_0$ *to* $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\delta_{\mathbf{Y}})$ *in* $\mathbf{rCat}_0$, *is a functor.*

Let $\mathbf{Y}$ be a restriction category, then by Lemma 3.2.3, $\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y}$ is stable meet semilattice fibration. If $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ is a map in $\mathbf{sFib}_0$, then by Lemma 3.2.5, there is a restriction functor $\mathcal{S}_s(F, F') : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathcal{S}_s(\partial_{\mathbf{Y}})$. But $\mathcal{S}_s(\partial_{\mathbf{Y}})$ and $\mathbf{Y}$ are different restriction categories in general. So a natural question is: *Is there a restriction functor from* $\mathcal{S}_s(\delta_{\mathbf{X}})$ *to* $\mathbf{Y}$*?* The answer is *Yes*!

Lemma 3.2.7 *Let* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *be a stable meet semilattice fibration and let* $\mathbf{Y}$ *be a restriction category. If* $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ *is a map in* $\mathbf{sFib}_0$, *then there is a restriction functor* $F^{\delta_{\mathbf{X}}} : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathbf{Y}$ *sending* $(f, \sigma) : A \to B$ *to* $(F(f))e_\sigma : F(A) \to F(B)$, *where the restriction idempotent* e_σ *is determined by* $F'(\sigma) = (F(A), e_\sigma) \in \mathbf{r}(\mathbf{Y})$.

For any object $A \in \mathbf{X}$ and any object $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$, $F'(\sigma) \in \mathbf{r}(\mathbf{Y})$ can be written as $(F(A), e_\sigma)$, where e_σ is a restriction idempotent over $F(A)$ in $\mathbf{Y}$. In order to prove Lemma 3.2.7, we need:

Lemma 3.2.8 *Let* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *be a stable meet semilattice fibration. Then*

(*i*) *For any object* $A \in \mathbf{X}$ *and any* $\sigma_1, \sigma_2 \in \delta_{\mathbf{X}}^{-1}(A)$,

$$e_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = 1_{F(A)} \text{ and } e_{\sigma_1 \wedge \sigma_2} = e_\sigma \wedge e_{\sigma_2};$$

(ii) For any map $f : A \to B$ in $\mathbf{X}$ and $\sigma \in \delta_{\mathbf{X}}^{-1}(B)$,

$$e_{f^*(\sigma)} = \overline{e_\sigma(F(f))};$$

(iii) For any map $(f,\sigma) : A \to B$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$, $e_\sigma = \overline{F(f)}e_\sigma$.

PROOF:

(*i*) By [**sfM.1**],

$$(F(A), e_{\top_{\delta_{\mathbf{X}}^{-1}(A)}}) = F'(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{Y}}^{-1}(F(A))} = (F(A), 1_{F(A)}).$$

Hence $e_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = 1_{F(A)}$. By [**sfM.2**],

$$\begin{aligned}(F(A), e_{\sigma_1\wedge\sigma_2}) &= F'(\sigma_1 \wedge \sigma_2)\\ &= F'(\sigma_1) \wedge F'(\sigma_2)\\ &= (F(A), e_{\sigma_1})(F(A), e_{\sigma_1})\\ &= (F(A), e_{\sigma_1}e_{\sigma_1}).\end{aligned}$$

Hence $e_{\sigma_1\wedge\sigma_2} = e_{\sigma_1}e_{\sigma_1}$.

(*ii*) By [**sfM.3**],

$$\begin{aligned}(F(A), e_{f^*(\sigma)}) &= F'(f^*(\sigma)) \\ &= (F(f))^*(F'(\sigma)) \\ &= (F(f))^*(F(B), e_\sigma) \\ &= (F(A), \overline{e_\sigma(F(f))}).\end{aligned}$$

Hence $e_{f^*(\sigma)} = \overline{e_\sigma(F(f))}$.

(*iii*) For any map $(f, \sigma) : A \to B$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$, since $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(A)})$, we have the following commutative diagram in $\tilde{\mathbf{X}}$:

$$\begin{array}{ccccc} & & f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) & & \\ & \overset{\leq}{\nearrow} & & \overset{\vartheta_f}{\searrow} & \\ \sigma & & \longrightarrow & & \top_{\delta_{\mathbf{X}}^{-1}(B)} \end{array}$$

where ϑ_f is the cartesian lifting of f at $\top_{\delta^{-1}(B)}$. Applying $\tilde{F}$, we have the following commutative diagram:

$$\begin{array}{ccccc} & & (F(A), e_{f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})}) & & \\ & \overset{1_{F(A)}}{\nearrow} & & \overset{F(f)}{\searrow} & \\ (F(A), e_\sigma) & & \xrightarrow{\quad F(f) \quad} & & (F(B), 1_{F(B)}) \end{array}$$

Hence $F(f) : (F(A), e_\sigma) \to (F(B), 1_{F(B)})$ is a map in $\mathbf{r}(\mathbf{Y})$ and therefore

$$e_\sigma = \overline{F(f)1_{F(B)}e_\sigma} = \overline{F(f)}e_\sigma.$$

□

PROOF OF LEMMA 3.2.7: Clearly, by [**sfM.1**],

$$F^{\delta_{\mathbf{X}}}(1_A) = F^{\delta_{\mathbf{X}}}(1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) = F(1_A) \cdot e_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = F(1_A) \cdot 1_{F(A)} = 1_{F(A)}.$$

For any maps $(f, \sigma) : A \to B$ and $(g, \sigma') : B \to C$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$, we have

$$\begin{aligned} F^{\delta_{\mathbf{X}}}((g,\sigma')(f,\sigma)) &= F^{\delta_{\mathbf{X}}}(gf, \sigma \wedge f^*(\sigma')) \\ &= F(gf) \cdot e_{\sigma \wedge f^*(\sigma')} \\ &= F(g) \cdot F(f) \cdot e_\sigma \wedge e_{f^*(\sigma')} \text{ (by Lemma 3.2.8 } (i)), \end{aligned}$$

and

$$\begin{aligned} F^{\delta_{\mathbf{X}}}(g,\sigma')F^{\delta_{\mathbf{X}}}(f,\sigma) &= (F(g) \cdot e_{\sigma'})(F(f) \cdot e_\sigma) \\ &= (F(g) \cdot F(f))\overline{e_\sigma(F(f))}e_\sigma \text{ (by [}\mathbf{R}.\mathbf{4}]) \\ &= (F(g) \cdot F(f))e_{f^*(\sigma')}e_\sigma \text{ (by Lemma 3.2.8 } (ii)). \end{aligned}$$

Hence

$$F^{\delta_{\mathbf{X}}}((g,\sigma')(f,\sigma)) = F^{\delta_{\mathbf{X}}}(g,\sigma')F^{\delta_{\mathbf{X}}}(f,\sigma).$$

Therefore $F^{\delta_{\mathbf{X}}}$ is a functor. Since

$$\begin{aligned}
F^{\delta_{\mathbf{X}}}(\overline{(f,\sigma)}) &= F^{\delta_{\mathbf{X}}}(1_A,\sigma) \\
&= F(1_A)\cdot e_\sigma \\
&= e_\sigma \\
&= \overline{F(f)}e_\sigma \text{ (by Lemma 3.2.8}(iii)) \\
&= \overline{(F(f))e_\sigma} \text{ (by [}\mathbf{R.3}]) \\
&= \overline{F^{\delta_{\mathbf{X}}}(f,\sigma)},
\end{aligned}$$

$F^{\delta_{\mathbf{X}}}$ is a restriction functor. □

Functor $\mathcal{R}_s : \mathbf{rCat}_0 \to \mathbf{sFib}_0$

If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then we have a functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ given by

$$\begin{array}{ccc}
(A, e_A) & \mapsto & (F(A), F(e_A)) \\
f\downarrow & \mapsto & \downarrow F(f) \\
(B, e_B) & \mapsto & (F(B), F(e_B))
\end{array}$$

and a commutative diagram

$$\begin{array}{ccc}
\mathbf{r}(\mathbf{X}) & \xrightarrow{\mathbf{r}(F)} & \mathbf{r}(\mathbf{Y}) \\
{\scriptstyle\partial_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle\partial_{\mathbf{Y}}} \\
\mathbf{X} & \xrightarrow{F} & \mathbf{Y}
\end{array}$$

For any map $f : A \to B$ in $\mathbf{X}$, we have

$$\begin{aligned}
\mathbf{r}(F)(\top_{\partial_{\mathbf{X}}^{-1}(A)}) &= \mathbf{r}(F)(A, 1_A) \\
&= (F(A), F(1_A)) \\
&= (F(A), 1_{F(A)}) \\
&= \top_{\partial_{\mathbf{Y}}^{-1}(F(A))},
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}(F)((B, e_B)(B, e'_B)) &= \mathbf{r}(F)(B, e_B e'_B) \\
&= (F(B), F(e_B) \cdot F(e'_B)) \\
&= (F(B), Fe_B)(F(B), F(e'_B)) \\
&= \mathbf{r}(F)(B, e_B) \cdot \mathbf{r}(F)(B, e'_B),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}(F)(f^*(B, e_B)) &= \mathbf{r}(F)(A, \overline{e_B f}) \\
&= (F(A), F(\overline{e_B f})) \\
&= (F(A), \overline{F(e_B) \cdot F(f)}) \\
&= (F(f))^*(F(B), F(e_B)) \\
&= (F(f))^*(\mathbf{r}(F)(B, e_B)).
\end{aligned}$$

Hence the conditions [**sfM.1**], [**sfM.2**], and [**sfM.3**] are satisfied and therefore

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$. We have more:

Lemma 3.2.9 *If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then there is a unique functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ such that*

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$.

PROOF: We already proved that

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$. To prove the uniqueness of $\mathbf{r}(F)$, assume that

$$(F, G) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$. Since

$$\begin{array}{ccc} \mathbf{r}(\mathbf{X}) & \xrightarrow{G} & \mathbf{r}(\mathbf{Y}) \\ {\scriptstyle \partial_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle \partial_{\mathbf{Y}}} \\ \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \end{array}$$

is commutative, G must map $f : (X_1, e_{X_1}) \to (X_2, e_{X_2})$ in $\mathbf{r}(\mathbf{X})$ to

$$F(f) : (F(X_1), i_{F(X_1)}) \to (F(X_2), i_{F(X_2)})$$

in $\mathbf{r}(\mathbf{Y})$, where $(F(X_k), i_{F(X_k)})$ is an idempotent over $F(X_k)$, $k = 1, 2$. By [**sfM.1**],

$$G(X_1, 1_{X_1}) = G(\top_{\partial_{\mathbf{X}}^{-1}(X_1)}) = \top_{\partial_{\mathbf{Y}}^{-1}(F(X_1))} = (F(X_1), 1_{F(X_1)}).$$

Note that $e_{X_1} : (X_1, e_{X_1}) \to (X_1, 1_{X_1})$ is a map in $\mathbf{r}(\mathbf{X})$. By [**sfM.3**],

$$\begin{aligned} G(X_1, e_{X_1}) &= G(e^*_{X_1}(X_1, 1_{X_1})) \\ &= (F(e_{X_1}))^*(G(X_1, 1_{X_1})) \\ &= (F(e_{X_1}))^*(F(X_1), 1_{F(X_1)}) \\ &= (F(X_1), \overline{1_{F(X_1)} F(e_{X_1})}) \\ &= (F(X_1), F(\overline{e_{X_1}})) \\ &= (F(X_1), F(e_{X_1})). \end{aligned}$$

Hence $G = \mathbf{r}(F)$, as desired. □

Lemma 3.2.10 $\mathcal{R}_s : \mathbf{rCat}_0 \to \mathbf{sFib}_0$, *taking* $F : \mathbf{X} \to \mathbf{Y}$ *in* $\mathbf{rCat}_0$ *to* $(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ *in* $\mathbf{sFib}_0$, *is a functor.*

PROOF: For any restriction functors $F : \mathbf{X} \to \mathbf{Y}$ and $G : \mathbf{Y} \to \mathbf{Z}$, we have

$$\mathcal{R}_s(GF) = (GF, \mathbf{r}(GF)) = (GF, \mathbf{r}(G)\mathbf{r}(F)) = (G, \mathbf{r}(G))(F, \mathbf{r}(F)) = \mathcal{R}_s(G)\mathcal{R}_s(F).$$

Clearly,

$$\mathcal{R}_s(1_{\mathbf{X}}) = (1_{\mathbf{X}}, 1_{\mathbf{r}(\mathbf{X})}).$$

Hence $\mathcal{R}_s$ is a functor. □

Adjunction $\mathcal{S}_s \dashv \mathcal{R}_s$

For a given stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, we can form $I_{\mathbf{X}} : \mathbf{X} \to \mathcal{S}_s(\delta_{\mathbf{X}})$ by sending $f : A \to B$ to $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$. Clearly, $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$ is a map in $\mathcal{S}_s(\delta_{\mathbf{X}})$ and so $I_{\mathbf{X}}$ is well-defined. For any maps $f : A \to B$ and $g : B \to C$ of $\mathbf{X}$, since $g^*(\top_{\mathbf{X}}^{-1}(C)) \leq \top_{\mathbf{X}}^{-1}(B)$,

$$
\begin{aligned}
I_{\mathbf{X}}(g)I_{\mathbf{X}}(f) &= (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}))(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \\
&= (gf, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \wedge f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, (gf)^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= I_{\mathbf{X}}(gf).
\end{aligned}
$$

Clearly, $I_{\mathbf{X}}(1_A) = (1_A, (1_A)^*(\top_{\delta_{\mathbf{X}}^{-1}(A)})) = (1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) = 1_{I_{\mathbf{X}}(A)}$. Hence $I_{\mathbf{X}}$ is a functor. Also, we can define

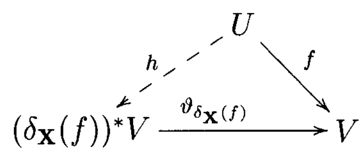
$$I_{\mathbf{X}}^{\delta_{\mathbf{X}}} : \tilde{\mathbf{X}} \to \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}}))$$

by

$$
\begin{array}{ccc}
U & \mapsto & (\delta_{\mathbf{X}}(U), (1_{\delta_{\mathbf{X}}(U)}, U)) \\
{\scriptstyle f}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle (\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}))} \\
V & \mapsto & (\delta_{\mathbf{X}}(V), (1_{\delta_{\mathbf{X}}(V)}, V))
\end{array}
$$

Since $1^*_{\delta_{\mathbf{X}}(U)}(U) = U \leq \top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(U))}$, $(1_{\delta_{\mathbf{X}}(U)}, U) : \delta_{\mathbf{X}}(U) \to \delta_{\mathbf{X}}(U)$ is a map in $\mathcal{S}_s(\delta_{\mathbf{X}})$ and so a restriction idempotent over $\delta_{\mathbf{X}}(U)$. Since there is a unique map $h : U \to$

$(\delta_{\mathbf{X}}(f))^*V$ such that $\vartheta_{\delta_{\mathbf{X}}(f)} h = f$:



$U \leq (\delta_{\mathbf{X}}(f))^*V$. Hence

$$\begin{aligned}
& \overline{(1_{\delta_{\mathbf{X}}(V)}, V)(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}))}(1_{\delta_{\mathbf{X}}(U)}, U) \\
= & \overline{(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}) \wedge (\delta_{\mathbf{X}}(f))^*V)}(1_{\delta_{\mathbf{X}}(U)}, U) \\
= & \overline{(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*V)}(1_{\delta_{\mathbf{X}}(U)}, U) \\
= & (1_{\delta_{\mathbf{X}}(U)}, (\delta_{\mathbf{X}}(f))^*V)(1_{\delta_{\mathbf{X}}(U)}, U) \\
= & (1_{\delta_{\mathbf{X}}(U)}, (\delta_{\mathbf{X}}(f))^*V \wedge U) \\
= & (1_{\delta_{\mathbf{X}}(U)}, U)
\end{aligned}$$

and therefore $(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))})) : (1_{\delta_{\mathbf{X}}(U)}, U) \to (1_{\delta_{\mathbf{X}}(V)}, V)$ is a map in $\mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}}))$. Then $I_{\mathbf{X}}^{\delta_{\mathbf{X}}}$ is well-defined.

Lemma 3.2.11 *If $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a stable meet semilattice fibration, then there is a functor $I_{\mathbf{X}} : \mathbf{X} \to \mathcal{S}_s(\delta_{\mathbf{X}})$ and a functor $I_{\mathbf{X}}^{\delta_{\mathbf{X}}} : \tilde{\mathbf{X}} \to \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}}))$. Moreover, $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ is a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\partial_{\mathcal{S}_s(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}})) \to \mathcal{S}_s(\delta_{\mathbf{X}}))$ in $\mathbf{sFib}_0$.*

Proof: We already showed that $I_{\mathbf{X}}$ is a functor and $I_{\mathbf{X}}^{\delta_{\mathbf{X}}}$ is well-defined. For any maps $f : U \to V$ and $g : V \to W$ in $\tilde{\mathbf{X}}$, since $(\delta_{\mathbf{X}} g)^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(W))}) \leq \top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}$, we have

$$
\begin{aligned}
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(g)I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(f) &= (\delta_{\mathbf{X}}g, (\delta_{\mathbf{X}}g)^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(W))}))(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))})) \\
&= (\delta_{\mathbf{X}}(gf), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}) \wedge (\delta_{\mathbf{X}}(f))^*(\delta_{\mathbf{X}}g)^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(W))})) \\
&= (\delta_{\mathbf{X}}(gf), (\delta_{\mathbf{X}}(f))^*(\delta_{\mathbf{X}}g)^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(W))})) \\
&= (\delta_{\mathbf{X}}(gf), (\delta_{\mathbf{X}}(gf))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(W))})) \\
&= I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(gf).
\end{aligned}
$$

Clearly,

$$
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(1_U) = (\delta_{\mathbf{X}}1_U, (\delta_{\mathbf{X}}1_U)^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(U))})) = (\delta_{\mathbf{X}}1_U, (\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(U))})) = 1_{I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(U)}.
$$

Hence $I_{\mathbf{X}}^{\delta_{\mathbf{X}}}$ is a functor. Obviously,

$$
\begin{CD}
\tilde{\mathbf{X}} @>{I_{\mathbf{X}}^{\delta_{\mathbf{X}}}}>> \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}})) \\
@V{\delta_{\mathbf{X}}}VV @VV{\partial_{\mathcal{S}_s(\delta_{\mathbf{X}})}}V \\
\mathbf{X} @>{I_{\mathbf{X}}}>> \mathcal{S}_s(\delta_{\mathbf{X}})
\end{CD}
$$

is commutative. For any map $f : A \to B$ in $\mathbf{X}$ and $\sigma, \sigma' \in \delta_{\mathbf{X}}^{-1}(B)$, we have

$$
\begin{aligned}
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\top_{\delta_{\mathbf{X}}^{-1}(A)}) &= (\delta_{\mathbf{X}}(\top_{\delta_{\mathbf{X}}^{-1}(A)}), (1_{\delta_{\mathbf{X}}(\top_{\delta_{\mathbf{X}}^{-1}(A)})}, \top_{\delta_{\mathbf{X}}^{-1}(A)})) \\
&= (A, (1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)})) \\
&= \top_{\partial_{\mathcal{S}_s(\delta_{\mathbf{X}})}^{-1}(I_{\mathbf{X}}(A))},
\end{aligned}
$$

$$\begin{aligned}
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\sigma \wedge \sigma') &= (\delta_{\mathbf{X}}(\sigma \wedge \sigma'), (1_{\delta_{\mathbf{X}}(\sigma\wedge\sigma')}, \sigma \wedge \sigma')) \\
&= (B, (1_B, \sigma \wedge \sigma')) \\
&= (B, (1_B, \sigma)) \wedge (B, (1_B, \sigma')) \\
&= (\delta_{\mathbf{X}}(\sigma), (1_{\delta_{\mathbf{X}}(\sigma)}, \sigma)) \wedge (\delta_{\mathbf{X}}(\sigma'), (1_{\delta_{\mathbf{X}}(\sigma')}, \sigma')) \\
&= I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\sigma) \wedge I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\sigma'),
\end{aligned}$$

and

$$\begin{aligned}
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(f^*(\sigma)) &= (\delta_{\mathbf{X}}(f^*(\sigma)), (1_{\delta_{\mathbf{X}}(f^*(\sigma))}, f^*(\sigma))) \\
&= (A, (1_A, f^*(\sigma))) \\
&= (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))^*(B, (1_B, \sigma)) \\
&= (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))^*(\delta_{\mathbf{X}}(\sigma), (1_{\delta_{\mathbf{X}}(\sigma)}, \sigma)) \\
&= (I_{\mathbf{X}}(f))^*(I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\sigma)).
\end{aligned}$$

Hence, conditions [**sfM.1**], [**sfM.2**], and [**sfM.3**] hold true, and therefore $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ is a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\partial_{\mathcal{S}_s(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}})) \to \mathcal{S}_s(\delta_{\mathbf{X}}))$ in $\mathbf{sFib}_0$. □

For any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, by Lemma 3.2.11, there exists a map $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathcal{S}_s(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_s(\delta_{\mathbf{X}})) \to \mathcal{S}_s(\delta_{\mathbf{X}}))$ in $\mathbf{sFib}_0$. This map turns out to be the unit of the adjunction $\mathcal{S}_s \dashv \mathcal{R}_s$. In fact, let $\mathbf{Y}$ be a restriction category and $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to \mathcal{R}_s(\mathbf{Y})$ any map in $\mathbf{sFib}_0$, by Lemma 3.2.7, there is a restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathbf{Y}$. It is easy to check that

$$(F^{\delta_{\mathbf{X}}}, \mathbf{r}(F^{\delta_{\mathbf{X}}}))(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) = (F, F').$$

If $G : \mathcal{S}_s(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to \mathbf{Y}$ is a restriction functor such that

$$(G, \mathbf{r}(G))(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) = (F, F'),$$

then $GI_{\mathbf{X}} = F$ and $\mathbf{r}(G)I_{\mathbf{X}}^{\delta_{\mathbf{X}}} = F'$. Hence for any map $f : A \to B$ in $\mathbf{X}$, G must map A to $F(A)$ and must map $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$ to $F(f) : F(A) \to F(B)$ and therefore $G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = F(f) = F^{\delta_{\mathbf{X}}}(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ since $e_{f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})} = \overline{F(f)}$. For any map $(f, \sigma) : A \to B$ in $\mathcal{S}_s(\delta_{\mathbf{X}})$,

$$\begin{aligned}
(F(A), e_\sigma) &= F'(\sigma) \\
&= \mathbf{r}(G)(I_{\mathbf{X}}^{\delta_{\mathbf{X}}})(\sigma) \\
&= \mathbf{r}(G)(A, (1_A, \sigma)) \\
&= (G(A), G(1_A, \sigma))
\end{aligned}$$

and so $G(1_A, \sigma) = e_\sigma$. Since $(f, \sigma) = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))(1_A, \sigma)$ and G is a restriction functor,

$$\begin{aligned}
G(f, \sigma) &= G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))G(1_A, \sigma) \\
&= F(f)e_\sigma \\
&= F^{\delta_{\mathbf{X}}}(f, \sigma).
\end{aligned}$$

Then $G = F^{\delta_{\mathbf{X}}}$ and so the uniqueness of $F^{\delta_{\mathbf{X}}}$ follows. Therefore, there is a unique restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}_s(\delta_{\mathbf{X}}) \to \mathbf{Y}$ such that

$$\begin{array}{ccc} (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) & \xrightarrow{(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})} & \mathcal{R}_s(\mathcal{S}_s(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})) \\ & \searrow^{(F,F')} & \downarrow \mathcal{R}_s(F^{\delta_{\mathbf{X}}}) \\ & & \mathcal{R}_s(\mathbf{Y}) \end{array} \qquad \begin{array}{c} \mathcal{S}_s(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \\ \exists! \downarrow F^{\delta_{\mathbf{X}}} \\ \mathbf{Y} \end{array}$$

commutes. Hence $\mathcal{S}_s \dashv \mathcal{R}_s$. Clearly, the counit ε of $\mathcal{S}_s \dashv \mathcal{R}_s$ is given by $\varepsilon_{\mathbf{C}} : \mathcal{S}_s(\mathcal{R}_s(\mathbf{C})) \to \mathbf{C}$ sending $(f, e_A) : A \to B$ to $fe_A : A \to B$, where e_A is a restriction idempotent such that $e_A \leq \overline{f}$, for each restriction category $\mathbf{C}$. We define $\lambda_{\mathbf{C}} : \mathbf{C} \to \mathcal{S}_s(\mathcal{R}_s(\mathbf{C}))$ by taking $f : A \to B$ to $(f, \overline{f}) : A \to B$. It is easy to check that $\lambda_{\mathbf{C}}$ is a functor such that $\varepsilon_{\mathbf{C}}\lambda_{\mathbf{C}} = 1_{\mathbf{C}}$ in $\mathbf{Cat}_0$. Hence $\varepsilon_{\mathbf{C}}$ is an epic in $\mathbf{rCat}_0$ and therefore is faithful. So, we proved:

Theorem 3.2.12 *There is an adjunction:*

$$\mathbf{rCat}_0 \underset{\mathcal{R}_s}{\overset{\mathcal{S}_s}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sFib}_0$$

with a faithful functor $\mathcal{R}_s$.

Remark For a given restriction category $\mathbf{C}$, the restriction functor

$$\varepsilon_{\mathbf{C}} : \mathcal{S}_s(\mathcal{R}_s(\mathbf{C})) \to \mathbf{C}$$

is faithful if and only if the restriction category $\mathbf{C}$ has the trivial restriction. To see this, we first suppose that $\varepsilon_{\mathbf{C}}$ is faithful. If $\mathbf{C}$ had a non-trivial restriction, then there would be a restriction idempotent $e_A \neq 1_A$ over A. But $\varepsilon_{\mathbf{C}}(1_A, e_A) = e_A =$

$\varepsilon_{\mathbf{C}}(e_A, e_A)$, which contradicts the fact that $\varepsilon_{\mathbf{C}}$ is faithful. Hence $\varepsilon_{\mathbf{C}}$ is faithful implies that $\mathbf{C}$ is with a trivial restriction. Conversely, if $\mathbf{C}$ is with the trivial restriction, then, clearly, $\mathcal{S}_s(\mathcal{R}_s(\mathbf{C})) = \mathbf{C}$.

Clearly, if $\mathbf{C}$ is a restriction category with the non-trivial restriction, then one has a sequence of restriction categories:

$$(\mathcal{S}_s\mathcal{R}_s)(\mathbf{C}), (\mathcal{S}_s\mathcal{R}_s)^2(\mathbf{C}), \cdots, (\mathcal{S}_s\mathcal{R}_s)^n(\mathbf{C}), \cdots.$$

3.2.4 The Image of $\mathcal{S}_s$: Fibered Restriction Categories

In the last subsection, we constructed the functor $\mathcal{S}_s : \mathbf{sFib}_0 \to \mathbf{rCat}_0$. The objective of this subsection is to characterize the class of restriction categories, which is the image of the functor $\mathcal{S}_s$. We call these categories fibered restriction categories. We begin with:

Lemma 3.2.13 *Let* $\mathbf{C}$ *be a restriction category. For any* $A, B \in \text{ob}\mathbf{C}$*,*

(*i*) $\text{map}_{\mathbf{C}}(A, B)$ *is a poset with the order given by*

$$f \leq g \Leftrightarrow f = g\overline{f};$$

(*ii*) $f \leq g$ *in* $\text{map}_{\mathbf{C}}(A, B)$ *implies* $\overline{f} \leq \overline{g}$ *in* $\text{map}_{\mathbf{C}}(A, A)$.

PROOF:

(*i*) Clearly, $f = f\overline{f}$ gives $f \leq f$.

If $f \leq g$ and $g \leq h$, then $f = g\overline{f}$ and $g = h\overline{g}$ and so, by [**R.3**],

$$f = g\overline{f} = h\overline{g}\,\overline{f} = h\overline{g\overline{f}} = h\overline{f}.$$

Hence $f \leq h$.

If $f \leq g$ and $g \leq f$, then $f = g\overline{f}$ and $g = f\overline{g}$ and so, by [**R.3**],

$$\overline{f} = \overline{g\overline{f}} = \overline{g}\,\overline{f} = \overline{f}\,\overline{g} = \overline{f\overline{g}} = \overline{g}.$$

Hence, by [**R.1**],

$$f = g\overline{f} = g\overline{g} = g,$$

as desired. Therefore, $\mathrm{map}_{\mathbf{C}}(A, B)$ is a poset.

(*ii*) $f \leq g$ gives $f = g\overline{f}$. Hence

$$\overline{f} = \overline{g\overline{f}} = \overline{g}\,\overline{f} = \overline{g}\,\overline{\overline{f}}$$

and therefore $\overline{f} \leq \overline{g}$.

□

Now, for a pair of objects A, B in a restriction category $\mathbf{C}$, we define

$$\mathrm{map}^{\max}_{\mathbf{C}}(A, B) = \{f \in \mathrm{map}_{\mathbf{C}}(A, B) \mid f \leq h \ \text{ implies } \ h = f \ \text{ in } \ \mathrm{map}_{\mathbf{C}}(A, B)\}.$$

A restriction category $\mathbf{C}$ is called *a fibered restriction category* if $\mathbf{C}$ satisfies the following two conditions:

[M.1] For any objects A, B and any $f \in \mathrm{map}_{\mathbf{C}}(A, B)$, there is a unique $m_f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$ such that $f \leq m_f$;

[M.2] For any objects A, B, C, $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$, and $g \in \mathrm{map}^{\max}_{\mathbf{C}}(B, C)$, $gf \in \mathrm{map}^{\max}_{\mathbf{C}}(A, C)$.

Note that for any map $f : A \to B$ in a restriction category $\mathbf{C}$, $m_{\overline{f}} = 1_A$ since $\overline{f} \leq 1_A$ and $1_A \leq g \Leftrightarrow g = 1_A$ in $\mathrm{map}_{\mathbf{C}}(A, A)$. Note also that $1_A \in \mathrm{map}^{\max}_{\mathbf{C}}(A, A)$ since $m_{1_A} = 1_A$.

For example, for any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, the restriction category $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a fibered restriction category. In fact, for any objects A, B,

$$\mathrm{map}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B) = \{(f, \sigma) | f \in \mathrm{map}_{\mathbf{X}}(A, B) \text{ and } \sigma \in \delta_{\mathbf{X}}^{-1}(A) \text{ with } \sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})\}.$$

Clearly, $(f, \sigma) \leq (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ since

$$(f, \sigma) = (f, \sigma)(1, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = (f, \sigma)\overline{(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))}.$$

If $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \leq (g, \sigma')$, then

$$\sigma' \leq g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$$

and

$$\begin{aligned}(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) &= (g, \sigma')\overline{(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))} \\ &= (g, \sigma')(1, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \\ &= (g, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \wedge \sigma').\end{aligned}$$

Hence $f = g$ and $f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \wedge \sigma' = f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$, and therefore $f = g$ and $\sigma' \geq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$. But $\sigma' \leq g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) = f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$. Then $\sigma' = f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$ and so $(g, \sigma') = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$. Therefore,

$$(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B)$$

and $(f, \sigma) \leq (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$. If $(g, \sigma') \in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B)$, then $\sigma' \leq g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$ and so $(g, \sigma') \leq (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$. Hence $(g, \sigma') = (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ and therefore

$$\mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B) = \{(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \mid f \in \mathrm{map}_{\mathbf{X}}(A, B)\}.$$

If $(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B)$ and $(f, \sigma) \leq (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$, then

$$(f, \sigma) = (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))\overline{(f, \sigma)} = (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))(1, \sigma) = (g, \sigma \wedge g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})),$$

and so $f = g$. Hence $(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ and therefore the uniqueness of $m_{(f,\sigma)} = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ follows. So [**M.1**] is satisfied.

$\forall (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, B)$, $(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(B, C)$, we have

$$\begin{aligned}
(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}))(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) &= (gf, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \wedge f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, (gf)^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&\in \mathrm{map}^{\max}_{\mathcal{S}_s(\delta_{\mathbf{X}})}(A, C).
\end{aligned}$$

Hence [**M.2**] is also satisfied. Thus, $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a fibered restriction category.

Let $\mathbf{C}$ be a fibered restriction category. We define $\mathbf{C}_{\max}$ by following data:

objects: the same as the objects of $\mathbf{C}$;

maps: for any objects A, B, $\mathrm{map}_{\mathbf{C}_{\max}}(A, B) = \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$;

composition: the same as in $\mathbf{C}$.

Then by [**M.2**], $\mathbf{C}_{\max}$ is a category. Now, we define $\tilde{\mathbf{C}}_{\max}$ to be the category given by

objects: (A, e_A), where e_A is a restriction idempotent over A in $\mathbf{C}$;

maps: a map f from (A, e_A) to (B, e_B) is a map $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$ such that $e_A = \overline{e_B f} e_A$;

composition: the same as in $\mathbf{C}$.

Obviously, there is a forgetful functor $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$, which forgets restriction idempotents.

Lemma 3.2.14 *For any fibered restriction category* $\mathbf{C}$*, the forgetful functor* $\partial_{\mathbf{C}_{\max}}$: $\tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ *is a stable meet semilattice fibration.*

PROOF: As in Lemma 3.2.3, for any map $f : A \to B$ in $\mathbf{C}_{\max}$ and any $(B, e_Y) \in \partial^{-1}_{\mathbf{C}_{\max}}(B)$, $f : (A, \overline{e_B f}) \to (B, e_B)$ is the cartesian lifting of a map $f : A \to B$ at (B, e_B). Note that each fiber

$$\partial^{-1}_{\mathbf{C}_{\max}}(A) = \{(A, e_A) \,|\, e_A : A \to A \text{ is a restriction idempotent on } A\}$$

is a meet semilattice with the order given by

$$(A, e_A) \leq (A, e'_A) \Leftrightarrow e_A = e'_A e_A,$$

with the binary meet given by

$$(A, e_A) \wedge (A, e'_A) = (A, e_A e'_A),$$

and with $(A, 1_A)$ as the top element. Obviously, for any map $f : A \to B$, $f^* : \partial^{-1}_{\mathbf{C}_{\max}}(B) \to \partial^{-1}_{\mathbf{C}_{\max}}(A)$, sending (B, e_B) to $(A, \overline{e_B f})$, is a stable meet semilattice homomorphism. Hence $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ is a stable meet semilattice fibration. □

Now, by Lemma 3.2.14, we can consider $\mathcal{S}_s(\partial_{\mathbf{C}_{\max}})$.

Proposition 3.2.15 *For any fibered restriction category* $\mathbf{C}$*,* $\mathcal{S}_s(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$.

PROOF: Define the functor $E : \mathcal{S}_s(\partial_{\mathbf{C}_{\max}}) \to \mathbf{C}$ by sending $(f, (A, e_A)) : A \to B$ to $f e_A : A \to B$. Then it is easy to check that E is a restriction functor.

For any map $f : A \to B$ in $\mathbf{C}$, by [**M.1**], there is a unique $m_f \in \text{map}_{\max}(A, B)$ such that $f \leq m_f$. Then $\overline{f} \leq \overline{m_f} = m_f^*(\top_{\partial^{-1}_{\mathbf{C}_{\max}}(B)})$ and so $(m_f, (A, \overline{f})) : A \to B$ is a map in $\mathcal{S}_s(\partial_{\mathbf{C}_{\max}})$. Now, we define $F : \mathbf{C} \to \mathcal{S}_s(\partial_{\mathbf{C}_{\max}})$ by sending $f : A \to B$ to $(m_f, (A, \overline{f})) : A \to B$. Clearly, F is well-defined and

$$F(1_A) = (m_{1_A}, (A, \overline{1_A})) = (1_A, (A, 1_A)) = 1_{F(A)}.$$

For any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$,

$$\begin{aligned}
F(g)F(f) &= (m_g, (B, \overline{g}))(m_f, (A, \overline{f})) \\
&= (m_g m_f, (A, \overline{f}) \wedge m_f^*(B, \overline{g})) \\
&= (m_{gf}, (A, \overline{f}\,\overline{\overline{g} m_f})) \\
&= (m_{gf}, (A, \overline{\overline{g} m_f \overline{f}})) \\
&= (m_{gf}, (A, \overline{\overline{g} f})) \;\; (\text{since } f \leq m_f \Rightarrow f = m_f \overline{f}) \\
&= (m_{gf}, (A, \overline{gf})) \\
&= F(gf).
\end{aligned}$$

Hence F is a functor. Note that

$$\begin{aligned}
F(\overline{f}) &= (m_{\overline{f}}, (A, \overline{\overline{f}})) \\
&= (1_A, (A, \overline{f})) \\
&= \overline{(m_{\overline{f}}, (A, \overline{f}))} \\
&= \overline{F(f)}.
\end{aligned}$$

Then F is a restriction functor.

For any map $f : A \to B$ in $\mathbf{C}$,

$$(EF)(f) = E(m_f, (A, \overline{f})) = m_f \overline{f} = f.$$

Hence $EF = 1_{\mathbf{C}}$.

On the other hand, for any map $(f, (A, \sigma_A)) : A \to B$ in $\mathcal{S}_s(\partial_{\mathbf{C}_{\max}})$,

$$\sigma_A \leq f^*(\top_{\partial^{-1}_{\mathbf{C}_{\max}}(B)}) = \overline{f}.$$

Then

$$\sigma_A = \overline{f}\sigma_A = \overline{fe_A}$$

and so $fe_A = f\overline{fe_A}$ which gives $fe_A \leq f$. Since $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$, we have $m_{fe_A} = f$. Hence

$$(FE)(f, (A, e_A)) = F(fe_A) = (m_{fe_A}, (A, \overline{fe_A})) = (f, (A, e_A)),$$

and therefore $FE = 1_{\mathcal{S}_s(\partial_{\mathbf{C}_{\max}})}$. Thus, $\mathcal{S}_s(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$. □

3.2.5 The Free Stable Meet Semilattice Fibrations

By Theorem 3.2.12 we have an adjunction $\mathcal{S}_s \dashv \mathcal{R}_s : \mathbf{rCat}_0 \to \mathbf{sFib}_0$. There is a *base functor* $U_f : \mathbf{sFib}_0 \to \mathbf{Cat}_0$, which sends $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ to $F : \mathbf{X} \to \mathbf{Y}$. One may ask whether the base functor U_f has a left adjoint. The objective of this subsection is to answer this question.

Categories Give Rise to Stable Meet Semilattice Fibrations

Each category gives rise to a stable meet semilattice fibration. In order to see this, we first recall Cockett-Lack's *closure operator* $\Downarrow(\ )$.

Let $\mathbf{C}$ be a category, $K = \{f_i : X \to Z_i \mid i \in I\}$ a set of maps with domain X, and $g : Y \to X$ a map. Then we write Kg for the set $\{f_i g \mid i \in I\}$, and $\Downarrow(K)$ for the set $\{f : X \to Z \mid uf = f_i \text{ for some } i \in I \text{ and some } u : Z \to Z_i\}$. Suppose that K and L are sets of maps with domain X. Clearly, if $K \subseteq L$ then $\Downarrow(K) \subseteq \Downarrow(L)$.

Lemma 3.2.16 *For any category, $\Downarrow(\_)$ is a Kuratowski closure operator on the maps with domain X. Namely, if K, K_1, and K_2 are sets of maps with domain X, then*

$$\Downarrow(\emptyset) = \emptyset, \Downarrow(K_1 \cup K_2) = \Downarrow(K_1) \cup \Downarrow(K_2), K \subseteq \Downarrow(K), \Downarrow(\Downarrow(K)) = \Downarrow(K).$$

PROOF: Clearly, $\Downarrow(\emptyset) = \emptyset$.

Since $\Downarrow(K_i) \subseteq \Downarrow(K_1 \cup K_2), i = 1, 2$, $\Downarrow(K_1) \cup \Downarrow(K_2) \subseteq \Downarrow(K_1 \cup K_2)$. On the other hand, for any $f \in \Downarrow(K_1 \cup K_2)$, there is a map u such that $uf \in K_1 \cup K_2$ and so $uf \in K_1$ or $uf \in K_2$. Hence $f \in \Downarrow(K_1) \cup \Downarrow(K_2)$ and therefore $\Downarrow(K_1 \cup K_2) \subseteq \Downarrow(K_1) \cup \Downarrow(K_2)$. Thus, $\Downarrow(K_1 \cup K_2) = \Downarrow(K_1) \cup \Downarrow(K_2)$.

For any $f \in K$, $1f = f \in K$. Hence $f \in \Downarrow(K)$ and therefore $K \subseteq \Downarrow(K)$.

Obviously, $\Downarrow(K) \subseteq \Downarrow(\Downarrow(K))$. For any $f \in \Downarrow(\Downarrow(K))$, there is a map u such that $uf \in \Downarrow(K)$ and so $vuf \in K$ for some map v. Then $f \in \Downarrow(K)$ and so $\Downarrow(\Downarrow(K)) \subseteq \Downarrow(K)$. Hence $\Downarrow(\Downarrow(K)) = \Downarrow(K)$. □

So, $\Downarrow(\_)$ endows $X/\mathbf{C}$ with a topology: the closed sets of this topology are the sets $\Downarrow(K)$ while the open sets (the complements of the closed sets) are *sieves* that are sets O such that $f \in O$ implies $uf \in O$. A map $f : X \to Y$ of $\mathbf{C}$ induces a map $\Downarrow(f)$ in the reverse direction between these topological spaces $Y/\mathbf{C}$ and $X/\mathbf{C}$:

$$\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}; h \mapsto hf.$$

Moreover, $\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}$ is a continuous map as shown in Lemma 3.2.18 below. We first observe the following simple topological result:

Lemma 3.2.17 *Let $f : T \to S$ be a map between two topological spaces T and S. Then the following are equivalent:*

(i) f is continuous;

(ii) For any closed set $F \subseteq S$, $f^{-1}(F)$ is closed in T;

(iii) For any set $A \subseteq T$, $f(\mathrm{cl}[A]) \subseteq \mathrm{cl}[f(A)]$;

(iv) For any set $A \subseteq T$, $\mathrm{cl}[f(A)] = \mathrm{cl}[f(\mathrm{cl}[A])]$.

PROOF:

$(i) \Leftrightarrow (ii)$: Since $f^{-1}(S \setminus F) = T \setminus f^{-1}(F)$.

$(ii) \Rightarrow (iii)$: Since $f(A) \subseteq \mathrm{cl}[f(A)]$, $A \subseteq f^{-1}(\mathrm{cl}[f(A)])$. By (ii), $f^{-1}(\mathrm{cl}[f(A)])$ is a closed set in T. So $\mathrm{cl}[A] \subseteq f^{-1}(\mathrm{cl}[f(A)])$. That is $f(\mathrm{cl}[A]) \subseteq \mathrm{cl}[f(A)]$.

$(iii) \Rightarrow (iv)$: For any set $A \subseteq T$, by (iii), $f(\mathrm{cl}[A]) \subseteq \mathrm{cl}[f(A)]$. Hence $\mathrm{cl}[f(\mathrm{cl}[A])] \subseteq \mathrm{cl}[f(A)]$. On the other hand, as $A \subset \mathrm{cl}[A]$, we have $\mathrm{cl}[f(A)] \subseteq \mathrm{cl}[f(\mathrm{cl}[A])]$ always. Therefore, $\mathrm{cl}[f(\mathrm{cl}[A])] = \mathrm{cl}[f(A)]$.

$(iv) \Rightarrow (ii)$: For any set $A \subseteq T$, by (iv), $\mathrm{cl}[f(\mathrm{cl}[A]] = \mathrm{cl}[f(A)]$. Then $f(\mathrm{cl}[A]) \subseteq \mathrm{cl}[f(\mathrm{cl}[A])] = \mathrm{cl}[f(A)]$. So, for any closed set $F \subseteq S$, we have

$$f(\mathrm{cl}[f^{-1}(F)]) \subseteq \mathrm{cl}[f(f^{-1}(F))] \subseteq \mathrm{cl}[F] = F.$$

Hence $\mathrm{cl}[f^{-1}(F)] \subseteq f^{-1}(F)$ and therefore $f^{-1}(F)$ is closed in T.

□

Lemma 3.2.18 *Let K, K' be sets of maps with domain X and f a map with codomain X. If $\Downarrow(K) = \Downarrow(K')$, then $\Downarrow(Kf) = \Downarrow(K'f)$. In particular,*

$$\Downarrow((\Downarrow(K))f) = \Downarrow(Kf)$$

and so $\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}$ is continuous.

Proof: If $x \in \Downarrow(Kf)$, then $ux = kf$ for some u and $k \in K$. As certainly $k \in \Downarrow(K)$, there is a map v such that $vk \in K'$. But then $vux = vkf$ so that $x \in \Downarrow(K'f)$. Hence $\Downarrow(Kf) \subseteq \Downarrow(K'f)$. Similarly, $\Downarrow(K'f) \subseteq \Downarrow(Kf)$. Therefore, $\Downarrow(Kf) = \Downarrow(K'f)$.

Since $\Downarrow(\Downarrow(K)) = \Downarrow(K)$, we have

$$\begin{aligned} \Downarrow(\Downarrow(f)(K)) &= \Downarrow(Kf) \\ &= \Downarrow((\Downarrow(K))f) \\ &= \Downarrow(\Downarrow(f)(\Downarrow(K))), \end{aligned}$$

for any $K \subseteq Y/\mathbf{C}$. By Lemma 3.2.17, $\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}$ is continuous. □

Now, we can form $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ by

$$\begin{array}{ccc} Y & \mapsto & Y/\mathbf{C} \\ {\scriptstyle f}\downarrow & \mapsto & \downarrow{\scriptstyle \Downarrow(f)} \\ X & \mapsto & X/\mathbf{C} \end{array}$$

By Lemma 3.2.18, we have the following property.

Proposition 3.2.19 $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ *is a functor.*

We also need the following lemma:

Lemma 3.2.20 *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and K a set of maps in $\mathbf{C}$ with domain X. Then $\Downarrow(F(K)) = \Downarrow(F(\Downarrow(K)))$.*

PROOF: Since $K \subseteq \Downarrow(K)$, $\Downarrow(F(K)) \subseteq \Downarrow(F(\Downarrow(K)))$. On the other hand, for any $x \in \Downarrow(F(\Downarrow K))$, $ux = Fy$ for some map u in $\mathbf{D}$ and some map $y \in \Downarrow(K)$. Since $y \in \Downarrow(K)$, $vy = k$ for some map $k \in K$ and some map v in $\mathbf{C}$. Then

$$(F(v) \cdot u)x = F(v) \cdot ux = F(v)F(y) = F(vy) = F(k) \in K,$$

and so $x \in \Downarrow(F(K))$. Hence, $\Downarrow(F(\Downarrow(K))) \subseteq \Downarrow(F(K))$. Therefore,

$$\Downarrow(F(K)) = \Downarrow(F(\Downarrow(K))).$$

□

Now, we form $\mathbf{s}(\mathbf{C})$ by the following data:

objects: $(X, \Downarrow\{x_1, \cdots, x_m\})$, where $X \in \mathrm{ob}(\mathbf{C})$ and $\{x_1, \cdots, x_m\} \subseteq \mathrm{ob}(X/\mathbf{C})$;

maps: a map from $(X, \Downarrow\{x_1, \cdots, x_m\})$ to $(Y, \Downarrow\{y_1, \cdots, y_n\})$ is a map $f : X \to Y$ in $\mathbf{C}$ such that $\Downarrow\{y_1 f, \cdots, y_n f\} \subseteq \Downarrow\{x_1, \cdots, x_m\}$;

composition and **identities** are formed as in $\mathbf{C}$.

By Lemma 3.2.18, $\mathbf{s}(\mathbf{C})$ is a category. Clearly, there is a forgetful functor $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$, which forgets the *sieves* $\Downarrow(K)$.

Lemma 3.2.21 *Let* $\mathbf{C}$ *be a category. Then the forgetful functor* $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ *is a stable meet semilattice fibration.*

Proof: For any map $f : X \to Y$ in $\mathbf{C}$ and any object $(Y, \Downarrow\{y_1, \cdots, y_n\}) \in \Delta_{\mathbf{C}}^{-1}(Y)$, clearly $f : (X, \Downarrow\{y_1 f, \cdots, y_n f\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$ is a map of $\mathbf{s}(\mathbf{C})$. Moreover,

$$f : (X, \Downarrow\{y_1 f, \cdots, y_n f\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$$

is the cartesian lifting of a map $f : X \to Y$ at $(Y, \Downarrow\{y_1, \cdots, y_n\})$. In fact, for any map $g : (Z, \Downarrow\{z_1, \cdots, z_k\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$ in $\mathbf{s}(\mathbf{C})$, we have

$$\Downarrow\{y_1 g, \cdots, y_n g\} \subseteq \Downarrow\{z_1, \cdots, z_k\}.$$

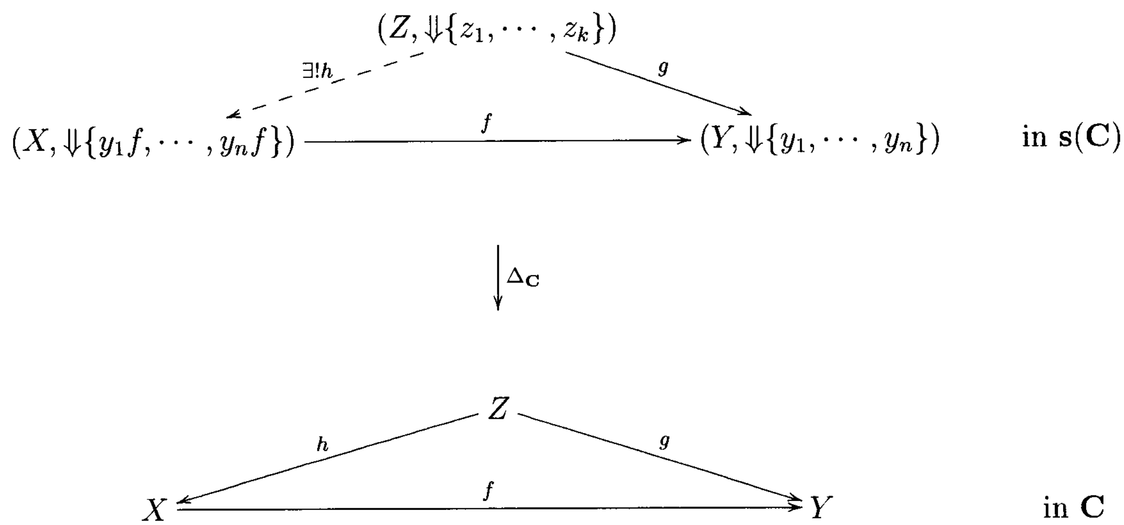
If $h : Z \to X$ is a map such that $fh = g$ in $\mathbf{C}$, then

$$\Downarrow\{y_1 fh, \cdots, y_n fh\} = \Downarrow\{y_1 g, \cdots, y_n g\} \subseteq \Downarrow\{z_1, \cdots, z_k\}.$$

Hence

$$h : (Z, \Downarrow\{z_1, \cdots, z_k\}) \to (X, \Downarrow\{y_1 f, \cdots, y_n f\})$$

is a map such that $\Delta_{\mathbf{C}}(h) = h$ and $fh = g$ in $\mathbf{s}(\mathbf{C})$:



The uniqueness of the map $h : (Z, \Downarrow\{z_1, \cdots, z_k\}) \to (X, \Downarrow\{y_1 f, \cdots, y_n f\})$ with the property that $fh = g$ in $\mathbf{s}(\mathbf{C})$ is obvious. Hence $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a fibration. Note that each fiber

$$\Delta_{\mathbf{C}}^{-1}(X) = \{(X, \Downarrow\{x_1, \cdots, x_m\}) \mid \{x_1, \cdots, x_m\} \subseteq \mathrm{map}(X/\mathbf{C})\}$$

is a meet semilattice with the order given by

$$(X, \Downarrow\{x_1, \cdots, x_m\}) \leq (X, \Downarrow\{x'_1, \cdots, x'_l\}) \Leftrightarrow \Downarrow\{x'_1, \cdots, x'_l\} \subseteq \Downarrow\{x_1, \cdots, x_m\},$$

with the binary meet given by

$$(X, \Downarrow\{x_1, \cdots, x_m\}) \wedge (X, \Downarrow\{x'_1, \cdots, x'_l\}) = (X, \Downarrow\{x_1, \cdots, x_m, x'_1, \cdots, x'_l\}),$$

and with $(X, \Downarrow\{1_X\})$ as the top element. Clearly, for any map $f : X \to Y$ in $\mathbf{C}$, $f^* : \Delta_{\mathbf{C}}^{-1}(Y) \to \Delta_{\mathbf{C}}^{-1}(X)$ which takes $(Y, \Downarrow\{y_1, \cdots, y_n\})$ to $(X, \Downarrow\{y_1 f, \cdots, y_n f\})$ is a stable meet semilattice homomorphism. Therefore, $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration. □

By Lemma 3.2.21, for any category $\mathbf{C}$, the forgetful functor $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration. If $F : \mathbf{X} \to \mathbf{Y}$ is a functor, then we have $\mathbf{s}(F) : \mathbf{s}(\mathbf{X}) \to \mathbf{s}(\mathbf{Y})$ defined by sending $f : (X, \Downarrow\{x_1, \cdots, x_m\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$ to $F(f) : (F(X), \Downarrow\{F(x_1), \cdots, F(x_m)\}) \to (F(Y), \Downarrow\{F(y_1), \cdots, F(y_n)\})$. By Lemma 3.2.20, $\mathbf{s}(F)$ is a well-defined functor.

For any map $f : A \to B$ in $\mathbf{X}$ and any $(B, \Downarrow\{b_1, \cdots, b_k\}), (B, \Downarrow\{b'_1, \cdots, b'_l\}) \in \Delta_{\mathbf{C}}^{-1}(B)$, we have:

$$\mathbf{s}(F)(\top_{\Delta_{\mathbf{Y}}^{-1}(A)}) = \mathbf{s}(F)(A, \Downarrow\{1_A\}) = (F(A), \Downarrow\{1_{F(A)}\}) = \top_{\Delta_{\mathbf{Y}}^{-1}(F(A))},$$

$$\begin{aligned} & \mathbf{s}(F)((B, \Downarrow\{b_1, \cdots, b_k\}) \wedge (B, \Downarrow\{b'_1, \cdots, b'_l\})) \\ = \; & \mathbf{s}(F)(B, \Downarrow\{b_1, \cdots, b_k, b'_1, \cdots, b'_l\}) \\ = \; & (F(B), \Downarrow\{F(b_1), \cdots, F(b_k), F(b'_1), \cdots, F(b'_l)\}) \\ = \; & \mathbf{s}(F)(B, \Downarrow\{b_1, \cdots, b_k\}) \wedge \mathbf{s}(F)(B, \Downarrow\{b'_1, \cdots, b'_l\}), \end{aligned}$$

and

$$\begin{aligned}
\mathbf{s}(F)(f^*(B, \Downarrow\{b_1, \cdots, b_k\})) &= \mathbf{s}(F)(A, \Downarrow\{b_1 f, \cdots, b_k f\}) \\
&= (F(A), \Downarrow\{F(b_1 f), \cdots, F(b_k f)\}) \\
&= (F(A), \Downarrow\{F(b_1)F(f), \cdots, F(b_k)F(f)\}) \\
&= (F(f))^*(F(B), \Downarrow\{F(b_1), \cdots, F(b_k)\}) \\
&= (F(f))^*(\mathbf{s}(F)(B, \Downarrow\{b_1, \cdots, b_k\})).
\end{aligned}$$

Hence $(F, \mathbf{s}(F))$ satisfies the conditions [**sfM.1**], [**sfM.2**], and [**sfM.3**] and therefore $(F, \mathbf{s}(F)) : \Delta_{\mathbf{X}} \to \Delta_{\mathbf{Y}}$ is a map in $\mathbf{sFib}_0$. Clearly, we have:

Lemma 3.2.22 *$F_f : \mathbf{Cat}_0 \to \mathbf{sFib}_0$, defined by sending $F : \mathbf{X} \to \mathbf{Y}$ to $(F, \mathbf{s}(F)) : \Delta_{\mathbf{X}} \to \Delta_{\mathbf{Y}}$, is a functor.*

In order to prove the universal property of $F_f \dashv U_f$, we also need:

Lemma 3.2.23 *For any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ and any functor $G : \mathbf{C} \to \mathbf{X}$, there is a unique functor $G' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ such that*

$$(G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$$

is a map in $\mathbf{sFib}_0$.

Proof: For any map $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ in $\mathbf{s}(\mathbf{C})$, where $c_i : C \to X_i$ and $d_j : D \to Y_j$ are maps in $\mathbf{C}$, $i = 1, \cdots, m, j = 1, \cdots, n$,

$$\Downarrow\{d_1 f, \cdots, d_n f\} \subseteq \Downarrow\{c_1, \cdots, c_m\}.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccc} & (C, \Downarrow\{c_1, \cdots, c_m\}) & \\ {\scriptstyle 1_C} \swarrow & & \searrow {\scriptstyle f} \\ (C, \Downarrow\{d_1 f, \cdots, d_n f\}) & \xrightarrow{\quad f \quad} & (D, \Downarrow\{d_1, \cdots, d_n\}) \end{array}$$

where $(C, \Downarrow\{d_1 f, \cdots, d_n f\}) = f^*(D, \Downarrow\{d_1, \cdots, d_n\})$. Now we define $G' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ by sending

$$f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$$

to

$$\vartheta \leq : \wedge_{i=1}^{m} (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))} \to \wedge_{j=1}^{n} (G(d_j))^* \top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))},$$

which is given by the following commutative diagram in $\tilde{\mathbf{X}}$:

$$\begin{array}{ccc} & \wedge_{i=1}^{m} (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))} & \\ {\scriptstyle \leq} \swarrow & & \searrow {\scriptstyle G'(f)} \\ (G(f))^* Y & \xrightarrow{\quad \vartheta_{G(f)} \quad} & Y \end{array}$$

where $Y = \wedge_{j=1}^{n} (G(d_j))^* \top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}$. Clearly,

$$G'(1_{(C, \Downarrow\{c_1, \cdots, c_m\})}) = G(1_C) = 1_{\wedge_{i=1}^{m} (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))}}.$$

For any map $g : (D, \Downarrow\{d_1, \cdots, d_n\}) \to (E, \Downarrow\{e_1, \cdots, e_k\})$ in $\mathbf{s}(\mathbf{C})$, where $e_i : E \to Z_i$ are maps in $\mathbf{C}$, $i = 1, \cdots, k$, since $G'(g)$ is given by the following commutative diagrams:

$$\begin{array}{ccc} & \wedge_{j=1}^{n}(G(d_j))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} & \\ {\scriptstyle\leq}\swarrow & & \searrow{\scriptstyle G'(g)} \\ (G(g))^*W & \xrightarrow{\quad\vartheta_{G(g)}\quad} & W \end{array}$$

and

$$\begin{array}{ccc} (G(f))^*(\wedge_{j=1}^{n}(G(d_j))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}) & \xrightarrow{\vartheta_{G(f)}} & \wedge_{j=1}^{n}(G(d_j))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} \\ {\scriptstyle\leq}\downarrow & & \downarrow{\scriptstyle\leq} \\ (G(f))^*(G(g))^*W & \xrightarrow{\quad\vartheta_{G(f)}\quad} & (G(g))^*W \end{array}$$

with $W = \wedge_{i=1}^{k}(G(e_i))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Z_i))}$, we have $G'(gf) = G'(g)G'(f)$. So $G' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ is a well-defined functor. It is routine to check that

$$\begin{array}{ccc} \mathbf{s}(\mathbf{C}) & \xrightarrow{G'} & \tilde{\mathbf{X}} \\ {\scriptstyle\Delta_{\mathbf{C}}}\downarrow & & \downarrow{\scriptstyle\delta_{\mathbf{X}}} \\ \mathbf{C} & \xrightarrow[G]{} & \mathbf{X} \end{array}$$

commutes. For any map $f : C \to D$ in $\mathbf{C}$ and any

$$(D, \Downarrow\{d_1, \cdots, d_n\}), (D, \Downarrow\{d'_1, \cdots, d'_k\}) \in \Delta_{\mathbf{C}}^{-1}(D),$$

where $d_i : D \to Y_i$ and $d'_j : D \to Y'_j$ are maps in $\mathbf{C}$, $i = 1, \cdots, n, j = 1, \cdots, k$, we have

$$G'(\top_{\Delta_{\mathbf{C}}^{-1}(C)}) = G'(C, \Downarrow\{1_C\}) = (G(1_C))^*(\top_{G(C)}) = \top_{G(C)},$$

$$\begin{aligned}
& G'((D, \Downarrow\{d_1, \cdots, d_n\}) \wedge (D, \Downarrow\{d'_1, \cdots, d'_k\})) \\
= \; & G'(D, \Downarrow\{d_1, \cdots, d_n, d'_1, \cdots, d'_k\}) \\
= \; & (\wedge_{i=1}^{n}(G(d_i))^*(\top_{G(Y_j)})) \wedge (\wedge_{j=1}^{k}(G(d'_i))^*(\top_{G(Y'_j)})) \\
= \; & G'(D, \Downarrow\{d_1, \cdots, d_n\}) \wedge G'(D, \Downarrow\{d'_1, \cdots, d'_k\}),
\end{aligned}$$

and

$$\begin{aligned}
G'(f^*(D, \Downarrow\{d_1, \cdots, d_n\})) &= G'(C, \Downarrow\{d_1 f, \cdots, d_n f\}) \\
&= \wedge_{i=1}^{n}(G(d_i f))^*(\top_{G(Y_i)}) \\
&= (G(f))^*(\wedge_{i=1}^{n}(G(d_i))^*(\top_{(Y_i)})) \\
&= (G(f))^*(G'(D, \Downarrow\{d_1, \cdots, d_n\})).
\end{aligned}$$

Hence $(G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ satisfies the conditions [**sfM**.**1**], [**sfM**.**2**], and [**sfM**.**3**] and therefore it is a map in $\mathbf{sFib}_0$.

Assume that $G'' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ is a functor such that

$$(G, G'') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$$

is a map in $\mathbf{sFib}_0$. Let $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ be a map in $\mathbf{s}(\mathbf{C})$. By [**sfM**.**1**],

$$G''(C, \Downarrow\{1_C\}) = G''(\top_{\Delta_{\mathbf{C}}^{-1}(C)}) = \top_{\delta_{\mathbf{X}}^{-1}(G(C))}.$$

By **[sfM.3]**,

$$\begin{aligned} G''(C, \Downarrow\{c_1\}) &= G''(c_1^*(X_1, \Downarrow\{1_{X_1}\})) \\ &= (G(c_1))^*(G''(X_1, \Downarrow\{1_{X_1}\})) \\ &= (G(c_1))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_1))}. \end{aligned}$$

By **[sfM.2]**,

$$\begin{aligned} G''(C, \Downarrow\{c_1, \cdots, c_m\}) &= G''(\wedge_{i=1}^m (C, \Downarrow\{c_i\})) \\ &= \wedge_{i=1}^m G''(C, \Downarrow\{c_i\}) \\ &= \wedge_{i=1}^m (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))}. \end{aligned}$$

Hence $G''(C, \Downarrow\{c_1, \cdots, c_m\}) = G'(C, \Downarrow\{c_1, \cdots, c_m\})$. Similarly,

$$G''(D, \Downarrow\{d_1, \cdots, d_n\}) = G'(D, \Downarrow\{d_1, \cdots, d_n\}).$$

Since

$$\begin{array}{ccc} \mathbf{s}(\mathbf{C}) & \xrightarrow{G''} & \tilde{\mathbf{X}} \\ {\scriptstyle \Delta_{\mathbf{C}}} \downarrow & & \downarrow {\scriptstyle \delta_{\mathbf{X}}} \\ \mathbf{C} & \xrightarrow[G]{} & \mathbf{X} \end{array}$$

commutes, for any map $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ in $\mathbf{s}(\mathbf{C})$, we have

$$\delta_{\mathbf{X}} G''(f) = G\Delta_{\mathbf{C}}(f) = G(f).$$

Hence there is a unique map

$$h : \wedge_{j=1}^{n}(G(d_j f))^{*}\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} \to \wedge_{j=1}^{n}(G(d_j f))^{*}\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}$$

in $\delta_{\mathbf{X}}^{-1}(G(C))$ such that $\delta_{\mathbf{X}}(h) = 1_{G(C)}$ and $\vartheta_{G(f)}h = G''(f)$:

$$\begin{array}{ccc} & \wedge_{i=1}^{m}(G(c_i))^{*}\top_{\delta_{\mathbf{X}}^{-1}(G(X_i))} & \\ \swarrow^{\exists ! h} & & \searrow^{G''(f)} \\ \wedge_{j=1}^{n}(G(d_j f))^{*}\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} & \xrightarrow{\vartheta_{G(f)}} & \wedge_{j=1}^{n}(G(d_j))^{*}\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} \end{array}$$

$$\Big\downarrow \delta_{\mathbf{X}}$$

$$\begin{array}{ccc} & G(C) & \\ \swarrow^{1_{G(C)}} & & \searrow^{G(f)} \\ G(C) & \xrightarrow{G(f)} & G(D) \end{array}$$

Since $\delta_{\mathbf{X}}$ is a stable meet semilattice fibration, $h = \leq$. Hence, by the definition of $G'(f)$,

$$G''(f) = \vartheta_{G(f)}h = \vartheta_{G(f)} \leq = G'(f)$$

and therefore the uniqueness of G' follows. □

Now we are ready to prove:

Theorem 3.2.24 *There is an adjunction:*

$$\mathbf{sFib}_0 \underset{U_f}{\overset{F_f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat}_0$$

with the identity unit $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$.

PROOF: For any category $\mathbf{C}$, clearly $U_f F_f(\mathbf{C}) = \mathbf{C}$. $\eta_{\mathbf{C}} = 1_{\mathbf{C}} : \mathbf{C} \to U_f F_f(\mathbf{C})$ turns out to be the unit of $F_f \dashv U_f$. In fact, for any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ and any functor $G : \mathbf{C} \to U_f(\delta_{\mathbf{C}})$, by Lemma 3.2.23 we have a unique map $G^* = (G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ in $\mathbf{sFib}_0$ such that

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{\eta_{\mathbf{C}}=1_{\mathbf{C}}} & U_f F_f(\mathbf{C}) & \qquad & F_f(\mathbf{C}) \\ & {\scriptstyle G}\searrow & \downarrow {\scriptstyle U_f(G^*)} & & \vdots\ {\scriptstyle \exists!\ G^*} \\ & & U_f(\delta_{\mathbf{X}}) & & \delta_{\mathbf{X}} \end{array}$$

commutes. Hence $F_f \dashv U_f$ with the unit $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$. □

So for any category $\mathbf{C}$, $F_f(\mathbf{C}) = \Delta_{\mathbf{C}}$ is the *free stable meet semilattice fibration* over $\mathbf{C}$.

3.2.6 The Free Restriction Categories over Categories

Recall that there is an evident forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$, which forgets restriction structures. One may ask:

Does U_r *have a left adjoint?*

In [7], Cockett and Lack gave an answer to the question by providing the free restriction categories over categories. In this subsection, as an application of Theorems 3.2.12 and 3.2.24, we shall revisit this question and reproduce Cockett-Lack's *free restriction category* $F_r(\mathbf{C})$ using the free stable meet semilattice fibration $\Delta_{\mathbf{C}}$ for any given category $\mathbf{C}$.

At first, we recall that adjunctions are composable.

Theorem 3.2.25 *Given two adjunctions:*

$$< F, G, \eta, \varepsilon >: \mathbf{C} \to \mathbf{X}, \;\; < F', G', \eta', \varepsilon' >: \mathbf{X} \to \mathbf{D},$$

the composition functors yields an adjunction

$$< F'F, GG', G\eta'F \cdot \eta, \varepsilon' \cdot F'\varepsilon G' >: \mathbf{C} \to \mathbf{D}.$$

Proof: See [17]. □

By Theorem 3.2.12, there is an adjunction $\mathbf{rCat}_0 \underset{\mathcal{R}_s}{\overset{\mathcal{S}_s}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{sFib}_0$ with a faithful functor $\mathcal{R}_s$. By Theorem 3.2.24, there is an adjunction $\mathbf{sFib}_0 \underset{U_f}{\overset{F_f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat}_0$ with the identity unit $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$. Now, we define F_r to be the functor $\mathcal{S}_s F_f : \mathbf{Cat}_0 \to \mathbf{rCat}_0$ and U_r to be the functor $U_f \mathcal{R}_s : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ which forgets the restriction structures. By Theorem 3.2.25, $F_r \dashv U_r$. Explicitly, for a given category $\mathbf{C}$, $F_r(\mathbf{C})$ is the category

- with the same objects as $\mathbf{C}$;
- with a map from C to D being a pair of $(f, \Downarrow(K))$, where $f : C \to D$ is a map of $\mathbf{C}$ and K is a set of maps in $\mathbf{C}$ with domain C such that

$$f \in \Downarrow(K) \text{ and } |K| < +\infty;$$

- with the composition given by

$$\begin{aligned}(g, \Downarrow(L))(f, \Downarrow(K)) &= (gf, \Downarrow(\Downarrow(K) \cup (\Downarrow f)\Downarrow(L))) \\ &= (gf, \Downarrow(K \cup Lf));\end{aligned}$$

- with the identities given by

$$1_C = (1_C, \Downarrow\{1_C\}).$$

So, $F_r(\mathbf{C})$ is Cockett-Lack's free restriction category over a given category $\mathbf{C}$.

If $F : \mathbf{C} \to \mathbf{D}$ is a functor, then the functor $F_r(F) : F_r(\mathbf{C}) \to F_r(\mathbf{D})$ is given by

$$\begin{array}{ccc} C_1 & \mapsto & F(C_1) \\ {\scriptstyle (f,\Downarrow K)}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle (F(f),\Downarrow(F(K)))} \\ C_2 & \mapsto & F(C_2) \end{array}$$

Furthermore, the functor $F_r : \mathbf{Cat}_0 \to \mathbf{rCat}_0$ is given by

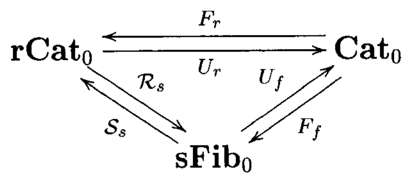
$$\begin{array}{ccc} \mathbf{C} & \mapsto & F_r(\mathbf{C}) \\ {\scriptstyle F}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle F_r(F)} \\ \mathbf{D} & \mapsto & F_r(\mathbf{D}) \end{array}$$

Clearly, we have:

Theorem 3.2.26 $F_r \dashv U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ *is an adjoint pair.*

Theorem 3.2.26, together with Theorems 3.2.12 and 3.2.24, immediately yields:

Theorem 3.2.27 *The following adjoint diagram*



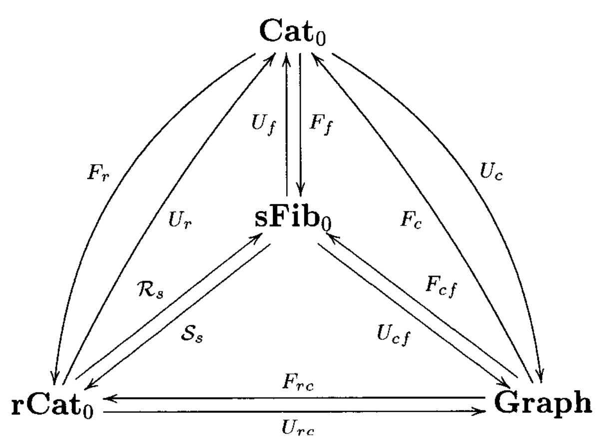
is commutative.

Recall that there is the *free category* generated by a graph [17]. So one has an adjunction:

$$\mathbf{Cat}_0 \underset{U_c}{\overset{F_c}{\underset{\perp}{\leftrightarrows}}} \mathbf{Graph}.$$

Define $U_{cf} = U_c U_f : \mathbf{sFib}_0 \to \mathbf{Graph}$ and $F_{cf} = F_f F_c : \mathbf{Graph} \to \mathbf{sFib}_0$ and define $U_{rc} = U_r U_c : \mathbf{Cat}_0 \to \mathbf{Graph}$ and $F_{rc} = F_c F_r : \mathbf{Graph} \to \mathbf{rCat}_0$. Then we have:

Theorem 3.2.28 *The following diagram of adjunctions*



is commutative.

3.3 Restriction Fibrations and Restriction Categories

In this section, we shall show that restriction categories and a certain class of fibrations (namely, restriction fibrations) are essentially the same.

3.3.1 Definition of Restriction Fibrations

Definition 3.3.1 *A fibration $\delta : \mathbf{D} \to \mathbf{C}$ is called a* restriction fibration *if for each object X of $\mathbf{C}$ the fiber $\delta^{-1}(X)$ is a meet semilattice in which $E_1 \leq E_2$ if and only if there is a map from E_1 to E_2 and for any object E of $\delta^{-1}(X)$ there is a map $\varepsilon_E : X \to X$ such that:*

[rF.1] $\varepsilon_{\top_{\delta^{-1}(X)}} = 1_X$,

[rF.2] $\varepsilon_E^*(\top_{\delta^{-1}(X)}) = E$, *where $\vartheta_{\varepsilon_E} : \varepsilon_E^*(\top_{\delta^{-1}(X)}) \to \top_{\delta^{-1}(X)}$ is the cartesian lifting of ε_E at $\top_{\delta^{-1}(X)}$,*

[rF.3] $\varepsilon_E \varepsilon_{E'} = \varepsilon_{E \wedge E'}$,

[rF.4] $\varepsilon_F(f) = f \varepsilon_{f^*(F)}$,

for any map $f : X \to Y$ in $\mathbf{C}$ and any $E, E' \in \delta^{-1}(X)$ and any $F \in \delta^{-1}(Y)$.

Fibrations and indexed categories are essentially the same. So we have the definition of restriction $\mathbf{C}$-indexed categories by translating that of restriction fibrations:

A $\mathbf{C}$-indexed category $(\,)^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{msLat}$ is called a *restriction $\mathbf{C}$-indexed category* if for any object X of $\mathbf{C}$ and any object E of X^* there is a map $\varepsilon_E : X \to X$ such that:

[rI.1] $\varepsilon_{\top_{X^*}} = 1_X$, [**rI.2**] $\varepsilon_E^*(\top_{X^*}) = E$,

[**rI.3**] $\varepsilon_E \varepsilon_{E'} = \varepsilon_{E \wedge E'}$,

[**rI.4**] $\varepsilon_F(f) = f\varepsilon_{f^*(F)}$,

for any map $f : X \to Y$ in $\mathbf{C}$ and any $E, E' \in X^*$ and any $F \in Y^*$.

For example, for any category $\mathbf{C}$, the identity fibration $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is a restriction fibration called the *trivial restriction fibration over* $\mathbf{C}$. Let $X^* = \{1_X\}$ and let $f^* : Y^* \to X^*$ be given by $f^*(1_Y) = 1_X$ for each map $f : X \to Y$ of $\mathbf{C}$. Then $()^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{msLat}$ is a restriction $\mathbf{C}$-indexed category called *the trivial restriction* $\mathbf{C}$*-indexed category.*

Lemma 3.3.2 *Let* $\mathbf{C}$ *be a restriction category. Then the forgetful functor* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *is a restriction fibration.*

PROOF: By the proof of Lemma 3.2.3, $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a fibration and each fiber

$$\partial_{\mathbf{C}}^{-1}(X) = \{(X, e_X) \mid e_X : X \to X \text{ is a restriction idempotent on } X\}$$

is a meet semilattice with the order given by

$$(X, e_X) \leq (X, e'_X) \Leftrightarrow e_X = e'_X e_X$$

which is equivalent to saying that there is a map from (X, e_X) to (X, e'_X) in $\partial_{\mathbf{C}}^{-1}(X)$, with the binary meet given by

$$(X, e_X) \wedge (X, e'_X) = (X, e_X e'_X),$$

and with $(X, 1_X)$ as the top element.

For any $X \in \mathbf{C}$ and any $(X, e_X) \in \partial_{\mathbf{C}}^{-1}(X)$, let $\varepsilon_{(X,e_X)} = e_X$. For any map $f : X \to Y$ in $\mathbf{C}$, and any $(X, e_X), (X, e'_X) \in \partial_{\mathbf{C}}^{-1}(X)$, and any $(Y, e_Y) \in \partial_{\mathbf{C}}^{-1}(Y)$, we have

$$\varepsilon_{\top_{\partial_{\mathbf{C}}^{-1}(X)}} = \varepsilon_{(X,1_X)} = 1_X,$$

$$\varepsilon^*_{(X,e_X)}(\top_{\partial_{\mathbf{C}}^{-1}(X)}) = e^*_X(X, 1_X) = (X, \overline{1_X e_X}) = (X, e_X),$$

and

$$\varepsilon_{(X,e_X)}\varepsilon_{(X,e'_X)} = e_X e'_X = \varepsilon_{(X,e_X)\wedge(X,e'_X)}.$$

So, [**rF**.**1**], [**rF**.**2**], and [**rF**.**3**] are satisfied.

On the other hand, since $f : (X, \overline{e_Y f}) \to (Y, e_Y)$ is the cartesian lifting of f at (Y, e_Y), we have

$$f^*(Y, e_Y) = (X, \overline{e_Y f}).$$

Then

$$f\varepsilon_{f^*(Y,e_Y)} = f\varepsilon_{(X,\overline{e_Y f})} = f\overline{e_Y f} = e_Y f = \varepsilon_{(Y,e_Y)} f.$$

Hence, [**rF**.**4**] is satisfied, too. Therefore, $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a restriction fibration. □

We shall see that every restriction fibration is of the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ for some restriction category $\mathbf{C}$ (see Proposition 3.3.10 below).

Each fiber of a restriction fibration has the following properties:

Lemma 3.3.3 *Let $\delta : \mathbf{D} \to \mathbf{C}$ be a restriction fibration and C an object in $\mathbf{C}$. Then*

(1) *For any object E of $\delta^{-1}(C)$, $\varepsilon_E^* = \_ \wedge E$;*

(2) *For any $E, E' \in \delta^{-1}(C)$, $E \leq E' \Leftrightarrow \varepsilon_E = \varepsilon_{E'}\varepsilon_E$;*

(3) *$\{\varepsilon_E \,|\, E \in \delta^{-1}(C)\}$ is a meet semilattice with the order given by $\varepsilon_E \leq \varepsilon_{E'} \Leftrightarrow \varepsilon_E = \varepsilon_E\varepsilon_{E'}$, with the binary meet given by $\varepsilon_E \wedge \varepsilon_{E'} = \varepsilon_E\varepsilon_{E'}$, and with $\varepsilon_{\top_{\delta^{-1}(C)}}$ as the top element. Moreover, $\{\varepsilon_E \,|\, E \in \delta^{-1}(C)\} \cong \delta^{-1}(C)$.*

PROOF: (1) Since for any $U \in X^*$,

$$\begin{aligned}
(\varepsilon_E^*)(U) &= (\varepsilon_E)^*(\varepsilon_U^*(\top_{X^*})) \text{ (by [}\mathbf{rI.2}\text{])} \\
&= (\varepsilon_U\varepsilon_E)^*(\top_{X^*}) \text{ (by functoriality of ( )}^*\text{)} \\
&= \varepsilon_{U\wedge E}^*(\top_{X^*}) \text{ (by [}\mathbf{rI.3}\text{])} \\
&= U \wedge E \text{ (by [}\mathbf{rI.2}\text{])},
\end{aligned}$$

we have

$$\varepsilon_E^* = \_ \wedge E.$$

(2) If $E \leq E'$, then $E = E \wedge E'$ and so $\varepsilon_E\varepsilon_{E'} = \varepsilon_{E\wedge E'} = \varepsilon_E$.

Conversely, if $\varepsilon_E = e_E\varepsilon_{E'}$, then $\varepsilon_E = \varepsilon_E\varepsilon_{E'} = \varepsilon_{E\wedge E'}$. Hence

$$E = \varepsilon_E^*(\top_{\delta^{-1}(X)}) = \varepsilon_{E\wedge E'}^*(\top_{\partial^{-1}(X)}) = E \wedge E',$$

and therefore $E \leq E'$.

(3) Clearly, $\{\varepsilon_E \,|\, E \in \delta^{-1}(C)\}$ is a meet semilattice and $\theta : \delta^{-1}(C) \to \{\varepsilon_E \,|\, E \in \delta^{-1}(C)\}$, given by $E \mapsto \varepsilon_E$, is a semilattice isomorphism. □

Obviously, one may give the corresponding result of Lemma 3.3.3 for a restriction $\mathbf{C}$-indexed category.

3.3.2 Characterization of Restriction Categories Using Fibrations

Restriction categories can be characterized by restriction fibrations as shown in the following theorem.

Theorem 3.3.4 *Let* $\mathbf{C}$ *be a category. Then the following are equivalent:*

(1) $\mathbf{C}$ *is a restriction category;*

(2) *There is a restriction fibration* $\partial : \mathbf{D} \to \mathbf{C}$*;*

(3) *There is a restriction* $\mathbf{C}$*-indexed category* $(\,)^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{msLat}$.

PROOF:

(1) $\Rightarrow$ (2) : If $\mathbf{C}$ is a restriction category, then by Lemma 3.3.2, we have a restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$.

(2) $\Rightarrow$ (3) : Each restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$ gives rise to a $\mathbf{C}$-indexed category

$$(\,)^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{StaLat}$$

by its inverse image functors:

$$\begin{array}{ccc} B & \mapsto & \delta^{-1}(B) \\ {\scriptstyle p}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle p^*} \\ E & \mapsto & \delta^{-1}(E) \end{array}$$

where $p^* : \delta^{-1}(B) \to \delta^{-1}(E)$ is given by

$$\begin{array}{ccc} (B, e_B) & \mapsto & (E, \overline{e_B p}) \\ f \downarrow & \mapsto & \downarrow p^* f \\ (B, e'_B) & \mapsto & (E, \overline{e'_B p}) \end{array}$$

It is easy to check that the induced **C**-indexed category $(\ )^*$ satisfies the conditions [**rI.1**], [**rI.2**], [**rI.3**], and [**rI.4**] so that it is a restriction **C**-indexed category.

(3) $\Rightarrow$ (1) : We define the restriction of a map $f : X \to Y$ in **C** by $\overline{f} = \varepsilon_{f^*(\top_{Y^*})}$. In order to prove that **C** is a restriction category with this restriction structure, we must check the four restriction axioms.

[**R.1**] For any map $f : X \to Y$, letting $F = \top_{Y^*}$ in [**rI.4**], by [**rI.1**] we have $f\varepsilon_{f^*(\top_{Y^*})} = \varepsilon_{\top_{Y^*}} f = f$. Then $f\overline{f} = f$.

[**R.2**] For maps $f : X \to Y_1, g : X \to Y_2$, by [**rI.3**],

$$\begin{aligned} \varepsilon_{f^*(\top_{Y_2^*})}\varepsilon_{g^*(\top_{Y_1^*})} &= \varepsilon_{f^*(\top_{Y_1^*}) \wedge g^*(\top_{Y_2^*})} \\ &= \varepsilon_{g^*(\top_{Y_2^*}) \wedge f^*(\top_{Y_1^*})} \\ &= \varepsilon_{g^*(\top_{Y_2^*})}\varepsilon_{f^*(\top_{Y_1^*})}. \end{aligned}$$

Therefore, $\overline{f}\overline{g} = \overline{g}\overline{f}$.

[R.3] For any maps $f : X \to Y_1, g : X \to Y_2$,

$$\begin{aligned}(g\varepsilon_{f^*(\top_{Y_1^*})})^*(\top_{Y_2^*}) &= (\varepsilon^*_{f^*(\top_{Y_1^*})}g^*)(\top_{Y_2^*})\\ &= \varepsilon^*_{f^*(\top_{Y_1^*})}(g^*(\top_{Y_2^*}))\\ &= f^*(\top_{Y_1^*}) \wedge g^*(\top_{Y_2^*}) \text{ (by Lemma 3.3.3 (1)).}\end{aligned}$$

Hence

$$\begin{aligned}\varepsilon_{(g\varepsilon_{f^*(\top_{Y_1^*})})^*(\top_{Y_2^*})} &= \varepsilon_{f^*(\top_{Y_1^*})\wedge g^*(\top_{Y_2^*})}\\ &= \varepsilon_{g^*(\top_{Y_2^*})\wedge f^*(\top_{Y_1^*})}\\ &= \varepsilon_{g^*(\top_{Y_2^*})}\varepsilon_{f^*(\top_{Y_1^*})} \text{ (by [rI.3]),}\end{aligned}$$

and therefore $\overline{g\overline{f}} = \overline{g}\overline{f}$.

[R.4] For any maps $f : X \to Y, g : Y \to Z$, by [**rI.4**],

$$\varepsilon_{g^*(\top_{Z^*})}f = f\varepsilon_{f^*(g^*(\top_{Z^*}))} = f\varepsilon_{(gf)^*(\top_{Z^*})}.$$

Hence $\overline{g}f = f\overline{gf}$.

□

3.3.3 The Category of Restriction Fibrations is Equivalent to $\mathbf{rCat}_0$

We shall form the category $\mathbf{rFib}_0$ of restriction fibrations first, then define the functors $\mathcal{R}_r : \mathbf{rCat}_0 \to \mathbf{rFib}_0$ and $\mathcal{E}_r : \mathbf{rFib}_0 \to \mathbf{rCat}_0$. Finally, we shall prove that $\mathcal{R}_r$ and $\mathcal{E}_r$ turn out to be an equivalence of categories.

The Category of Restriction Fibrations $\mathbf{rFib}_0$ and Functors $\mathcal{R}_r$ and $\mathcal{E}_r$

Let $\mathbf{rFib}_0$ be the category with

objects: restriction fibrations: $\delta : \mathbf{D} \to \mathbf{C}$;

maps: a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\delta' : \mathbf{D}' \to \mathbf{C}')$ is a pair (F, F'), where $F : \mathbf{C} \to \mathbf{C}'$ and $F' : \mathbf{D} \to \mathbf{D}'$ are functors such that

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F'} & \mathbf{D}' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \mathbf{C} & \xrightarrow{F} & \mathbf{C}' \end{array}$$

commutes and for any map $f : X \to Y$ in $\mathbf{C}$ and any $E \in \delta^{-1}(X), W \in \delta^{-1}(Y)$, the following conditions are satisfied:

[pR.1] $F'(\top_{\delta^{-1}(X)}) = \top_{(\delta')^{-1}(F(X))}$,

[pR.2] $F(\varepsilon_E) = \varepsilon_{F'(E)}$,

[pR.3] $F'(f^*(W)) = (F(f))^*(F'(W))$;

composition and **identities** are defined by: $(F_2, F_2')(F_1, F_1') = (F_2F_1, F_2'F_1')$, and $1_{(\delta:\mathbf{D}\to\mathbf{C})} = (1_\mathbf{C}, 1_\mathbf{D})$.

By Lemma 3.3.2, each restriction category $\mathbf{C}$ gives rise to a restriction fibration $(\partial_\mathbf{C} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$, denoted by $\mathcal{R}_r(\mathbf{C})$. If $F : \mathbf{D} \to \mathbf{C}$ is a restriction functor, then we have a (restriction) functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{D}) \to \mathbf{r}(\mathbf{C})$ given by

$$\begin{array}{ccc} (D_1, e_1) & \mapsto & (F(D_1), F(e_1)) \\ {\scriptstyle d}\downarrow & \mapsto & \downarrow{\scriptstyle F(d)} \\ (D_2, e_2) & \mapsto & (F(D_2), F(e_2)) \end{array}$$

It is easy to check that

$$\begin{array}{ccc} \mathbf{r}(\mathbf{D}) & \xrightarrow{\mathbf{r}(F)} & \mathbf{r}(\mathbf{C}) \\ {\scriptstyle \partial_{\mathbf{D}}}\downarrow & & \downarrow{\scriptstyle \partial_{\mathbf{C}}} \\ \mathbf{D} & \xrightarrow{F} & \mathbf{C} \end{array}$$

commutes. For any map $d : X \to Y$ in $\mathbf{D}$, $(X, \varepsilon_E) \in \partial_{\mathbf{D}}^{-1}(X)$, and $(Y, \varepsilon_W) \in \partial_{\mathbf{D}}^{-1}(Y)$, we have

$$\mathbf{r}(F)(\top_{\partial_{\mathbf{D}}^{-1}(X)}) = \mathbf{r}(F)(X, 1_X) = (F(X), 1_{F(X)}) = \top_{\partial_{\mathbf{C}}^{-1}(F(X))},$$

$$F\varepsilon_{(X,\varepsilon_E)} = F\varepsilon_E = \varepsilon_{(F(X),F\varepsilon_E)} = \varepsilon_{\mathbf{r}(F)(X,\varepsilon_E)},$$

and

$$\begin{aligned} \mathbf{r}(F)f^*(Y, \varepsilon_W) &= \mathbf{r}(F)(X, \overline{\varepsilon_W f}) \\ &= (F(X), F(\overline{\varepsilon_W f})) \\ &= (F(X), \overline{F(\varepsilon_W f)}) \\ &= (F(X), \overline{(F(\varepsilon_W))(F(f))}) \\ &= (F(f))^*(F(Y), F(\varepsilon_W)) \\ &= (F(f))^*(\mathbf{r}(F)(Y, \varepsilon_W)). \end{aligned}$$

Hence $\mathcal{R}_r(F) = (\mathbf{r}(F), F)$ satisfies [**pR**.**1**], [**pR**.**2**], and [**pR**.**3**], and therefore $\mathcal{R}_r(F) : \mathcal{R}_r(\mathbf{D}) \to \mathcal{R}_r(\mathbf{C})$ is actually a map in $\mathbf{rFib}_0$. Now, clearly, $\mathcal{R}_r : \mathbf{rCat}_0 \to \mathbf{rFib}_0$

given by

$$\begin{array}{ccc} \mathbf{D} & \mapsto & \mathcal{R}_r(\mathbf{D}) = (\partial_{\mathbf{D}} : \mathbf{r}(\mathbf{D}) \to \mathbf{D}) \\ {\scriptstyle F}\downarrow & \mapsto & \downarrow{\scriptstyle \mathcal{R}_r(F)} \\ \mathbf{C} & \mapsto & \mathcal{R}_r(\mathbf{C}) = (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}) \end{array}$$

is a functor. So we already proved:

Lemma 3.3.5 $\mathcal{R}_r : \mathbf{rCat}_0 \to \mathbf{rFib}_0$, *taking* $F : \mathbf{D} \to \mathbf{C}$ *to* $\mathcal{R}_r(F) : (\partial_{\mathbf{D}} : \mathbf{r}(\mathbf{D}) \to \mathbf{D}) \to (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$, *is a functor.*

In the inverse direction, we have:

Lemma 3.3.6 *There is a functor* $\mathcal{E}_r : \mathbf{rFib}_0 \to \mathbf{rCat}_0$ *sending* $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ *to* $F : \mathbf{C} \to \mathbf{C}'$, *where* $\mathbf{C}$ *and* $\mathbf{C}'$ *are the restriction categories with the reduced restriction structures by* δ *and* δ', *respectively.*

PROOF: For any map $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ in $\mathbf{rFib}_0$, since (F, F') satisfies [**pR**.**1**], [**pR**.**2**], and [**pR**.**3**], we have

$$\begin{aligned} F(\overline{f}) &= F(\varepsilon_{f^*(\top_{\delta^{-1}(Y)})}) \\ &= \varepsilon_{F'(f^*(\top_{\delta^{-1}(Y)}))} \text{ (by [}\mathbf{pR.2}\text{])} \\ &= \varepsilon_{(F(f))^*(F'(\top_{\delta^{-1}(Y)}))} \text{ (by [}\mathbf{pR.3}\text{])} \\ &= \varepsilon_{(F(f))^*(\top_{(\delta')^{-1}(FY)})} \text{ (by [}\mathbf{pR.1}\text{])} \\ &= \overline{F(f)}. \end{aligned}$$

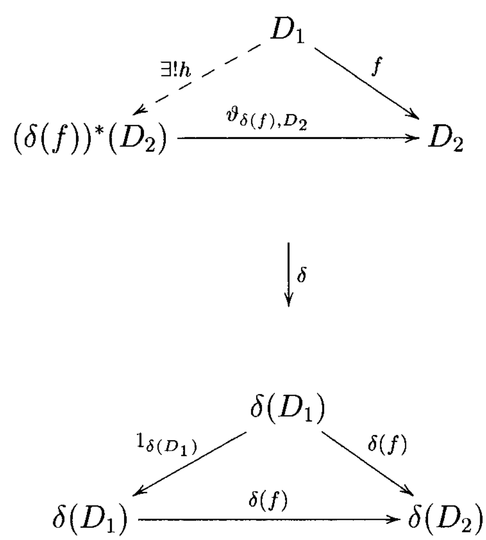
Hence $F : \mathbf{C} \to \mathbf{C}'$ is a restriction functor and therefore $\mathcal{E}_r$ is well-defined. The functoriality of $\mathcal{E}_r$ is obvious. So $\mathcal{E}_r$ is a functor. □

Adjunction $\mathcal{E}_r \dashv \mathcal{R}_r$

Given any restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, $\mathbf{C}$ is a restriction category with the restriction structure induced by δ. Assume that $F : \mathbf{C} \to \mathbf{E}$ is a restriction functor. In order to prove $\mathcal{E}_r \dashv \mathcal{R}_r$, we construct $F_\delta : \mathbf{D} \to \mathbf{r}(\mathbf{E})$ by

$$\begin{array}{ccc} D_1 & \mapsto & ((F\delta)(D_1), F(\varepsilon_{D_1})) \\ {\scriptstyle f}\downarrow & \mapsto & \downarrow{\scriptstyle (F\delta)(f)} \\ D_2 & \mapsto & ((F\delta)(D_2), F(\varepsilon_{D_2})) \end{array}$$

Since δ is a restriction fibration, for a given map $f : D_1 \to D_2$ in $\mathbf{D}$ there is a unique map $h : D_1 \to (\delta(f))^*(D_2)$ in $\delta^{-1}(\delta(D_1))$ such that $\vartheta_{\delta(f),D_2} h = f$:



Then $D_1 \leq (\delta(f))^*(D_2)$ in $\delta^{-1}(\delta(D_1))$ and so

$$\begin{aligned}
\overline{F(\varepsilon_{D_2})(F\delta)(f)}F(\varepsilon_{D_1}) &= \overline{F(\varepsilon_{D_2}\delta(f))}F(\varepsilon_{D_1}) \\
&= F(\overline{\varepsilon_{D_2}\delta(f)})F(\varepsilon_{D_1}) \\
&= F(\varepsilon_{(\varepsilon_{D_2}(\delta f))^*(\top_{\delta^{-1}(\delta D_2)})})F(\varepsilon_{D_1}) \\
&= F(\varepsilon_{(\delta f)^*\varepsilon^*_{D_2}(\top_{\delta^{-1}(\delta D_2)})})F(\varepsilon_{D_1}) \\
&= F(\varepsilon_{(\delta f)^*(D_2)})F(\varepsilon_{D_1}) \text{ (by [}\mathbf{rF.2}\text{])} \\
&= F(\varepsilon_{(\delta f)^*D_2}\varepsilon_{D_1}) \\
&= F(\varepsilon_{(\delta f)^*(D_2)\wedge D_1}) \text{ (by [}\mathbf{rF.3}\text{])} \\
&= F(\varepsilon_{D_1}).
\end{aligned}$$

Hence

$$(F\delta)(f) : ((F\delta)(D_1), F(\varepsilon_{D_1})) \to ((F\delta)(D_2), F(\varepsilon_{D_2}))$$

is a map in $\mathbf{r}(\mathbf{E})$ and therefore F_δ is well-defined. Clearly, F_δ is a functor such that

$$\begin{array}{ccc}
\mathbf{D} & \xrightarrow{F_\delta} & \mathbf{r}(\mathbf{E}) \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\partial_{\mathbf{E}}} \\
\mathbf{C} & \xrightarrow{F} & \mathbf{E}
\end{array}$$

commutes.

For any map $f : X \to Y$ in $\mathbf{C}$ and any $E \in \delta^{-1}(X), W \in \delta^{-1}(Y)$, we have

$$
\begin{aligned}
F_\delta(\top_{\delta^{-1}(X)}) &= ((F\delta)(\top_{\delta^{-1}(X)}), F(\varepsilon_{\top_{\delta^{-1}(X)}})) \\
&= (F(X), F(1_X)) \text{ (by } [\mathbf{rF.1}]) \\
&= (F(X), 1_{F(X)}) \\
&= \top_{(\partial_{\mathbf{E}})^{-1}(F(X))},
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{F_\delta(E)} &= \varepsilon_{(F_\delta(E), F(\varepsilon_E))} \\
&= \varepsilon_{(F(X), F(\varepsilon_E))} \\
&= F(\varepsilon_E),
\end{aligned}
$$

and

$$
\begin{aligned}
F_\delta(f^*(W)) &= ((F\delta)(f^*W), F(\varepsilon_{f^*W})) \\
&= (F(X), F(\varepsilon_{f^*W})) \\
&= (F(X), F(\varepsilon_{(\varepsilon_W f)^*(\top_{\delta^{-1}(Y)})})) \\
&= (F(X), F(\overline{\varepsilon_W f})) \\
&= (F(X), \overline{(F(\varepsilon_W))(F(f))}) \\
&= (F(f))^*(F(Y), F(\varepsilon_W)) \\
&= (F(f))^*((F\delta)(W), F(\varepsilon_W)) \\
&= (F(f))^*(F_\delta(W)).
\end{aligned}
$$

Hence [**pR.1**], [**pR.2**], [**pR.3**] are satisfied. Therefore (F, F_δ) is a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\partial_\mathbf{E} : \mathbf{r}(\mathbf{E}) \to \mathbf{E})$ in $\mathbf{rFib}_0$. But we have more:

Lemma 3.3.7 *Let $\delta : \mathbf{D} \to \mathbf{C}$ be a restriction fibration and $\mathbf{C}$ the restriction category with the restriction structure induced by δ. Then (F, G) is a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\partial_\mathbf{E} : \mathbf{r}(\mathbf{E}) \to \mathbf{E})$ in $\mathbf{rFib}_0$ if and only if $G = F_\delta$. In particular, if $F : \mathbf{C} \to \mathbf{D}$ is a restriction functor, then (F, G) is a map from $(\partial_\mathbf{C} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ to $(\partial_\mathbf{D} : \mathbf{r}(\mathbf{D}) \to \mathbf{D})$ in $\mathbf{rFib}_0$ if and only if $G = \mathbf{r}(F)$.*

PROOF: We already saw that $(F, F_\delta) : (\delta : \mathbf{D} \to \mathbf{C}) \to (\partial_\mathbf{C} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ is a map in $\mathbf{rFib}_0$. So we need to prove the uniqueness of F_δ. Assume that $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\partial_\mathbf{C} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ is a map in $\mathbf{rFib}_0$. Then

$$
\begin{array}{ccc}
\mathbf{D} & \xrightarrow{F'} & \mathbf{r}(\mathbf{E}) \\
{\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \partial_\mathbf{E}} \\
\mathbf{C} & \xrightarrow{F} & \mathbf{E}
\end{array}
$$

commutes and so F' must send any map $f : D_1 \to D_2$ in $\mathbf{D}$ to

$$(F\delta)(f) : ((F\delta)(D_1), e_{(F\delta)(D_1)}) \to ((F\delta)(D_2), e_{(F\delta)(D_2)}),$$

where $e_{(F\delta)(D_i)}$ are restriction idempotents over $(F\delta)(D_i)$ in $\mathbf{E}$, $i = 1, 2$. By [**pR.2**],

$$e_{(F\delta)(D_1)} = \varepsilon_{F'(D_1)} = F(\varepsilon_{D_1}).$$

Similarly,

$$e_{(F\delta)(D_2)} = \varepsilon_{F'(D_2)} = F(\varepsilon_{D_2}).$$

So $F' = F_\delta$ and the uniqueness of F_δ follows, as desired. Now, by applying the restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ to δ, $\mathbf{r}(F) : \mathbf{r}(\mathbf{C}) \to \mathbf{r}(\mathbf{D})$ is the unique functor such that $(F, \mathbf{r}(F)) : (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}) \to (\partial_{\mathbf{D}} : \mathbf{r}(\mathbf{D}) \to \mathbf{D})$ is a map in $\mathbf{rFib}_0$. □

For any restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, by Lemma 3.3.7 there exists a map $(1_{\mathbf{C}}, 1_\delta) : (\delta : \mathbf{D} \to \mathbf{C}) \to (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ taking $f : D_1 \to D_2$ to $\delta(f) : (\delta(D_1), \varepsilon_{D_1}) \to (\delta(D_2), \varepsilon_{D_2})$ in $\mathbf{rFib}_0$. This map turns out to be the unit of the adjunction $\mathcal{E}_r \dashv \mathcal{R}_r$. In fact, let $\mathbf{E}$ be a restriction category and $(F, G) : (\delta : \mathbf{D} \to \mathbf{C}) \to \mathcal{R}_r(\mathbf{E})$ any map in $\mathbf{rFib}_0$. By Lemma 3.3.7, G must be F_δ and so there exists a unique map $F : \mathcal{E}_r(\delta : \mathbf{D} \to \mathbf{C}) \to \mathbf{E}$ in $\mathbf{rCat}_0$ such that

$$\begin{array}{ccc} (\delta : \mathbf{D} \to \mathbf{C}) & \xrightarrow{(1_{\mathbf{C}}, 1_\delta)} & \mathcal{R}_r\mathcal{E}_r(\delta : \mathbf{D} \to \mathbf{C}) \\ & \searrow^{(F,G)} & \downarrow{\scriptstyle \mathcal{R}_r(F)} \\ & & \mathcal{R}_r(\mathbf{E}) \end{array} \qquad \begin{array}{c} \mathcal{E}_r(\delta : \mathbf{D} \to \mathbf{C}) \\ {\scriptstyle \exists!}\,\vdots\,{\scriptstyle F} \\ \mathbf{E} \end{array}$$

commutes. Hence $\mathcal{E}_r \dashv \mathcal{R}_r$. Clearly, $\mathcal{E}_r\mathcal{R}_r = 1_{\mathbf{rCat}_0}$. Therefore, we proved:

Lemma 3.3.8 *There is an adjunction:*

$$\mathbf{rCat}_0 \underset{\mathcal{R}_r}{\overset{\mathcal{E}_r}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{rFib}_0$$

with identity counit so that $\mathcal{R}_r$ is full and faithful.

$\mathbf{rCat}_0$ and $\mathbf{rFib}_0$ are Equivalent

By Lemma 3.3.2, each restriction category $\mathbf{C}$ gives rise to a restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$. Naturally, we want to know whether all restriction fibrations are of the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$, namely, whether $\mathcal{R}_r$ is surjective on objects. If so, by

Lemma 3.3.8, $\mathcal{R}_r$ is an equivalence of categories.

Lemma 3.3.9 *If $\delta : \mathbf{D} \to \mathbf{C}$ is a restriction fibration and $\mathbf{C}$ is the restriction category with the restriction induced by δ, then there is a map $(1_{\mathbf{C}}, G)$ from $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ to $\delta : \mathbf{D} \to \mathbf{C}$ in $\mathbf{rFib}_0$.*

PROOF: Note that $\mathbf{r}(\mathbf{C}) = \{(C, \varepsilon_E) \,|\, \forall C \in \text{ob}\mathbf{C}, E \in \delta^{-1}(C)\}$. If $f : (C_1, \varepsilon_{E_1}) \to (C_2, \varepsilon_{E_2})$ is a map in $\mathbf{r}(\mathbf{C})$, then

$$\begin{aligned} \varepsilon_{E_1} &= \overline{\varepsilon_{E_2} f}\varepsilon_{E_1} \\ &= \varepsilon_{(\varepsilon_{E_2} f)^*(\top_{C_2^*})}\varepsilon_{E_1} \\ &= \varepsilon_{f^*\varepsilon^*_{E_2}(\top_{C_2^*})}\varepsilon_{E_1} \\ &= \varepsilon_{f^*E_2}\varepsilon_{E_1}. \end{aligned}$$

Hence $\varepsilon_{E_1} \leq \varepsilon_{f^*E_2}$ and therefore $E_1 \leq f^*E_2$ by Lemma 3.3.3. Then there is a unique map $\text{leq} : E_1 \to f^*E_2$ in $\delta^{-1}(C_1)$ satisfying $\delta(\text{leq}) = 1_{C_1}$ and so a map $\text{lift}(f) = \vartheta_{f,E_2}\text{leq} : E_1 \to E_2$ satisfying $\delta(\text{lift}(f)) = f$, where $\vartheta_{f,E_2} : f^*E_2 \to E_2$ is the cartesian lifting of f at E_2. Now, we can define $G : \mathbf{r}(\mathbf{C}) \to \mathbf{D}$ by

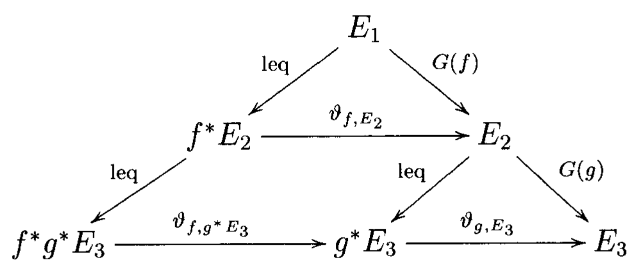
$$\begin{array}{ccc} (C_1, \varepsilon_{E_1}) & \mapsto & E_1 \\ f\big\downarrow & \mapsto & \big\downarrow \text{lift}(f) \\ (C_2, \varepsilon_{E_2}) & \mapsto & E_2 \end{array}$$

By Lemma 3.3.3, $\varepsilon_{E_1} = \varepsilon_{E_2} \Leftrightarrow E_1 = E_2$. Clearly, $\text{lift}(f)$ is determined uniquely by

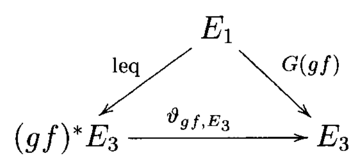
f. So G is well-defined. By the functoriality of $(\ )^*$, $1^*_{(C_1,\varepsilon_{E_1})}(E_1) = E_1$ and so

$$G(1_{(C_1,\varepsilon_{E_1})}) = \text{lift}(1_{(C_1,\varepsilon_{E_1})}) = 1_{E_1} = 1_{G(C_1,\varepsilon_{E_1})}.$$

On the other hand, for any maps $f : (C_1, \varepsilon_{E_1}) \to (C_2, \varepsilon_{E_2})$ and $g : (C_2, \varepsilon_{E_2}) \to (C_3, \varepsilon_{E_3})$ in $\mathbf{r}(\mathbf{C})$, by the definition of cartesian lifting we have the following commutative diagrams:



and



Hence $G(gf) = G(g)G(f)$. Therefore, G is a functor. For any map $f : X \to Y$ in $\mathbf{C}$, $(X, \varepsilon_E) \in \partial_{\mathbf{C}}^{-1}(X)$, and $(Y, \varepsilon_W) \in \partial_{\mathbf{C}}^{-1}(Y)$,

$$G(X, 1_X) = G(X, \varepsilon_{\top_{\delta^{-1}(X)}}) = \top_{\partial^{-1}(X)} = \top_{\delta^{-1}(1_{\mathbf{C}}(X))},$$

$$1_{\mathbf{C}}\varepsilon_E = \varepsilon_E = \varepsilon_{G(X,\varepsilon_E)},$$

and

$$\begin{aligned} G(f)^*(Y, \varepsilon_W) &= G(X, \overline{\varepsilon_W f}) \\ &= G(X, \varepsilon_{(\varepsilon_W f)^*(\top_{\delta^{-1}(Y)})}) \\ &= G(X, \varepsilon_{f^*W}) \\ &= f^*W \\ &= (1_{\mathbf{C}}(f))^*(G(Y, \varepsilon_W)). \end{aligned}$$

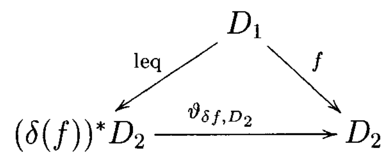
Hence $(1_{\mathbf{C}}, G)$ satisfies [**pR**.1], [**pR**.2], and [**pR**.3] and therefore $(1_{\mathbf{C}}, G)$ is a map from $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ to $\delta : \mathbf{D} \to \mathbf{C}$ in $\mathbf{rFib}_0$. □

Proposition 3.3.10 *Each restriction fibration can be written in the form* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *for some restriction category* $\mathbf{C}$. *More precisely, if* $\delta : \mathbf{D} \to \mathbf{C}$ *is a restriction fibration then* $(\delta : \mathbf{D} \to \mathbf{C}) \cong (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ *in* $\mathbf{rFib}_0$.

PROOF: Assume that $\delta : \mathbf{D} \to \mathbf{C}$ is a restriction fibration. By Theorem 3.3.4, we have a restriction category $\mathbf{C}$ with the restriction structure induced by δ and a restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$. By Lemma 3.3.7, there is a map $(1_{\mathbf{C}}, 1_\delta) : (\delta : \mathbf{D} \to \mathbf{C}) \to \mathcal{R}_r(\mathbf{C})$ in $\mathbf{rFib}_0$. By Lemma 3.3.9, there is a map $(1_{\mathbf{C}}, G)$ from $(\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ to $(\delta : \mathbf{D} \to \mathbf{C})$ in $\mathbf{rFib}_0$. Clearly, $(1_{\mathbf{C}}, 1_\delta)(1_{\mathbf{C}}, G) = 1_{\mathcal{R}_r(\mathbf{C})}$. For any map $d : D_1 \to D_2$ in $\mathbf{D}$, we have

$$(1_{\mathbf{C}}, 1_\delta)(d : D_1 \to D_2) = (\delta(d) : (\delta(D_1), \varepsilon_{D_1}) \to (\delta(D_2), \varepsilon_{D_2})).$$

But $\operatorname{lift}(\delta d) = d$ since there is a unique map $\operatorname{leq} : D_1 \to (\delta f)^* D_2$ such that



commutes. Hence

$$\begin{aligned}(1_{\mathbf{C}}, G)(1_{\mathbf{C}}, 1_\delta)(d : D_1 \to D_2) &= (1_{\mathbf{C}}, G)(\delta(d) : (\delta(D_1), \varepsilon_{D_1}) \to (\delta(D_2), \varepsilon_{D_2})) \\ &= (\operatorname{lift}(\delta d) : D_1 \to D_2) \\ &= (d : D_1 \to D_2),\end{aligned}$$

and therefore

$$(1_{\mathbf{C}}, G)(1_{\mathbf{C}}, 1_\delta) = 1_{(\delta : \mathbf{D} \to \mathbf{C})}.$$

Thus,

$$(\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}) \cong (\delta : \mathbf{D} \to \mathbf{C}).$$

□

Now we are ready to prove:

Theorem 3.3.11 *There is an adjoint equivalence:*

$$\mathbf{rCat}_0 \underset{\mathcal{R}_r}{\overset{\mathcal{E}_r}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{rFib}_0$$

with identity counit.

Proof: Combining Lemma 3.3.8 and Proposition 3.3.10, $\mathcal{R}_r$ is not only full and faithful, but also surjective on objects. Hence $\mathcal{R}_r$ is an equivalence of categories. □

Remark. By Theorem 3.2.12, we have an adjunction $\mathbf{rCat}_0 \underset{\mathcal{R}_s}{\overset{\mathcal{S}_s}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{sFib}_0$. Clearly, $\mathcal{S}_s \dashv \mathcal{R}_s$ and $\mathcal{E}_r \dashv \mathcal{R}_r$ are connected since

$$\mathcal{R}_r(\mathbf{rCat}_0) \approx \mathbf{rFib}_0 \hookrightarrow \mathbf{sFib}_0.$$

Chapter 4

Range Restriction Categories and Fibrations

In this chapter, we shall introduce the notion of range stable meet semilattice fibrations and show that such fibrations produce range restriction structures. We shall also introduce the notion of range restriction fibrations and show that those fibrations are the same as range restriction categories.

4.1 Range Stable Meet Semilattice Fibrations and Range Restriction Categories

In Section 3.2, we showed that there is an adjunction between $\mathbf{rCat}_0$ and $\mathbf{sFib}_0$. In this section, we shall provide the analogous result for range restriction categories.

4.1.1 Range Stable Meet Semilattice Fibrations

Definition 4.1.1 A range stable meet semilattice fibration *is a stable meet semilattice fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{Category}$ *in which for any map* $f : A \to B$ *in* $\mathbf{X}$ *there is a monotone map* $f_! : \delta_{\mathbf{X}}^{-1}(A) \to \delta_{\mathbf{X}}^{-1}(B)$ *such that* $(gf)_! = g_! f_!$ *for any map* $g : B \to C$ *in* $\mathbf{X}$ *and for any* $\sigma_A \in \delta_{\mathbf{X}}^{-1}(A), \sigma_B \in \delta_{\mathbf{X}}^{-1}(B)$, *the following conditions are satisfied:*

[rsF.1] $(f_!(\sigma_A)) \wedge \sigma_B \leq f_!(\sigma_A \wedge f^*(\sigma_B))$,

[rsF.2] $\sigma_A \wedge (f^*(\sigma_B)) \leq f^*(f_!(\sigma_A) \wedge \sigma_B)$,

[rsF.3] $f_! f^*(\sigma_B) \leq \sigma_B$.

For example, for any category $\mathbf{C}$, the identity fibration $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is a range stable meet semilattice fibration.

Lemma 4.1.2 *If* $\mathbf{C}$ *is a range restriction category, then the forgetful functor* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *is a range stable meet semilattice fibration.*

PROOF: By Lemma 3.2.3, $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ a stable meet semilattice fibration. For any map $f : A \to B$ in $\mathbf{C}$ and any $(A, e_A) \in \partial_{\mathbf{C}}^{-1}(A)$, we define $f_! : \partial_{\mathbf{C}}^{-1}(A) \to \partial_{\mathbf{C}}^{-1}(B)$ by sending (A, e_A) to $(B, \widehat{fe_A})$. If $(A, e_A) \leq (A, e'_A)$ in $\partial_{\mathbf{C}}^{-1}(A)$, then $e_A = e_A e'_A = e'_A e_A$ and so

$$\widehat{fe'_A}\widehat{fe_A} = \widehat{fe_A e'_A}\widehat{fe_A} = \widehat{fe_A e'_A} = \widehat{fe_A}.$$

Hence $\widehat{fe_A} \leq \widehat{fe'_A}$ and therefore

$$f_!(A, e_A) = (B, \widehat{fe_A}) \leq (B, \widehat{fe'_A}) = f_!(A, e'_A).$$

Therefore $f_!$ is monotone. For any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$ and $(A, e_A) \in \partial_{\mathbf{C}}^{-1}(A)$, by [**RR**.**4**] we have

$$g_! f_!(A, e_A) = g_!(B, \widehat{fe_A}) = (C, \widehat{g\widehat{fe_A}}) = (C, \widehat{gfe_A}) = (gf)_!(A, e_A).$$

Hence $(gf)_! = g_! f_!$. So it suffices to check the conditions [**rsF**.**1**], [**rsF**.**2**], and [**rsF**.**3**]. Let $f : A \to B$ be any map in $\mathbf{C}$ and $(A, e_A) \in \partial_{\mathbf{C}}^{-1}(A), (B, e_B) \in \partial_{\mathbf{C}}^{-1}(B)$.

[rsF.1] Since

$$
\begin{aligned}
\widehat{f e_A \overline{e_B f}} &= (f e_A \widehat{\overline{e_B f}} e_A) \\
&= (f e_A \widehat{\overline{e_B f e_A}}) \text{ (by } [\mathbf{R.3}]) \\
&= \widehat{\overline{e_B} f e_A} \text{ (by } [\mathbf{R.4}]) \\
&= \widehat{f e_A} e_B \text{ (by } [\mathbf{RR.3}]),
\end{aligned}
$$

we have $(f_!(A, e_A)) \wedge (B, e_B) \leq f_!((A, e_A) \wedge f^*(B, e_B))$.

[rsF.2] Since

$$
\begin{aligned}
\overline{e_B \widehat{f e_A} f}\, \overline{e_B f} e_A &= \overline{e_B \widehat{f e_A} f} e_A \overline{e_B f} \\
&= \overline{e_B \widehat{f e_A} f e_A}\, \overline{e_B f} \text{ (by } [\mathbf{R.3}]) \\
&= \overline{e_B f e_A}\, \overline{e_B f} \text{ (by } [\mathbf{RR.2}]) \\
&= \overline{e_B f e_A \overline{e_B f}} \text{ (by } [\mathbf{R.3}]) \\
&= \overline{e_B f \overline{e_B f} e_A} \\
&= \overline{e_B f e_A} \text{ (by } [\mathbf{R.1}]) \\
&= \overline{e_B f} e_A \text{ (by } [\mathbf{R.3}]),
\end{aligned}
$$

we have $\overline{e_B f} e_A \leq \overline{e_B \widehat{f e_A} f}$. Hence

$$(A, e_A) \wedge (f^*(B, e_B)) \leq f^*(f_!(A, e_A) \wedge (B, e_B)).$$

[rsF.3] Since

$$
\begin{aligned}
\widehat{f\overline{e_B f}}\, e_B &= \widehat{\overline{e_B} f}\, e_B \text{ (by [}\mathbf{R}.\mathbf{4}\text{])} \\
&= \overline{e_B}\, \widehat{\overline{e_B} f} \\
&= \widehat{\overline{e_B} e_B f} \text{ (by [}\mathbf{RR}.\mathbf{3}\text{])} \\
&= \widehat{\overline{e_B} f} \\
&= \widehat{f\overline{e_B f}} \text{ (by [}\mathbf{R}.\mathbf{4}\text{])},
\end{aligned}
$$

we have $\widehat{f\overline{e_B f}} \leq e_B$. Hence $f_! f^*(B, e_B) \leq (B, e_B)$.

□

Suppose that $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a range stable meet semilattice fibration. Then by Proposition 3.2.4, we have the restriction category $\mathcal{S}_s(\delta_{\mathbf{X}})$ with the restriction given by $\overline{(f, \sigma)} = (1_A, \sigma)$ for any map $(f, \sigma) : A \to B$. It becomes a range restriction category if we define $\widehat{(f, \sigma)} = (1_B, f_!(\sigma))$ for any map $(f, \sigma) : A \to B$. We need the following lemma:

Lemma 4.1.3 *Let $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ be a range stable meet semilattice fibration and $f : A \to B$ a map in $\mathbf{X}$. Then*

(1) *If $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$ is such that $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$, then $\sigma \leq f^* f_!(\sigma)$;*

(2) *If $\sigma_A \in \delta_{\mathbf{X}}^{-1}(A)$ and $\sigma_B \in \delta_{\mathbf{X}}^{-1}(B)$, then $f_!(\sigma_A \wedge f^*(\sigma_B)) = f_!(\sigma_A) \wedge \sigma_B$.*

PROOF:

(1) Since $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$, by [**rsF.2**], we have

$$\sigma = \sigma \wedge f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \leq f^*(f_!(\sigma) \wedge \top_{\delta_{\mathbf{X}}^{-1}(B)}) = f^* f_!(\sigma).$$

(2) Since $\sigma_A \wedge f^*(\sigma_B) \leq \sigma_A$, $\sigma_A \wedge f^*(\sigma_B) \leq f^*(\sigma_B)$ and $f_!$ is monotone,

$$f_!(\sigma_A \wedge f^*(\sigma_B)) \leq f_!(\sigma_A),$$

and by [**rsF.3**],

$$f_!(\sigma_A \wedge f^*(\sigma_B)) \leq f_! f^*(\sigma_B) \leq \sigma_B.$$

Then

$$f_!(\sigma_A \wedge f^*(\sigma_B)) \leq f_!(\sigma_A) \wedge \sigma_B.$$

On the other hand, by [**rsF.1**],

$$f_!(\sigma_A) \wedge \sigma_B \leq f_!(\sigma_A \wedge f^*(\sigma_B)).$$

Hence $f_!(\sigma_A \wedge f^*(\sigma_B)) = f_!(\sigma_A) \wedge \sigma_B$.

□

Proposition 4.1.4 *If $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a range stable meet semilattice fibration, then $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a range restriction category with range and restriction given by $\overline{(f, \sigma)} = (1_A, \sigma)$ and $\widehat{(f, \sigma)} = (1_B, f_!(\sigma))$ for any map $(f, \sigma) : A \to B$. Denote this range restriction category by $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$.*

PROOF: By Proposition 3.2.4, $\mathcal{S}_s(\delta_{\mathbf{X}})$ is a restriction category. So it suffices to check the four range axioms. Let $(f,\sigma) : A \to B$, $(g,\sigma') : B \to C$ be maps in $\mathcal{S}_s(\delta_{\mathbf{X}})$.

[RR.1] $\overline{\widehat{(f,\sigma)}} = \overline{(1_B, f_!(\sigma))} = (1_B, f_!(\sigma)) = \widehat{(f,\sigma)}$.

[RR.2] Since $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$, by Lemma 4.1.3 (1), $\sigma \leq f^* f_!(\sigma)$. Hence

$$\widehat{(f,\sigma)}(f,\sigma) = (1_B, f_!(\sigma)(f,\sigma)) = (f, \sigma \wedge f^* f_!(\sigma)) = (f,\sigma).$$

[RR.3] By Lemma 4.1.3 (2),

$$f_!(\sigma \wedge f^*(\sigma')) = f_!(\sigma) \wedge \sigma'.$$

Hence

$$\begin{aligned}
\widehat{\overline{(g,\sigma')}(f,\sigma)} &= \widehat{(1_B,\sigma')(f,\sigma)} \\
&= \widehat{(f,\sigma \wedge f^*(\sigma'))} \\
&= (1_B, f_!(\sigma \wedge f^*(\sigma'))) \\
&= (1_B, f_!(\sigma) \wedge \sigma') \\
&= (1_B, \sigma')(1_B, f_!(\sigma)) \\
&= \overline{(g,\sigma')}\,\widehat{(f,\sigma)},
\end{aligned}$$

and therefore

$$\widehat{\overline{(g,\sigma')}(f,\sigma)} = \overline{(g,\sigma')}\,\widehat{(f,\sigma)}.$$

[RR.4] Again, by Lemma 4.1.3 (2),

$$f_!(\sigma \wedge f^*(\sigma')) = f_!(\sigma) \wedge \sigma'.$$

Hence

$$\begin{aligned}
\widehat{(g,\sigma')\widehat{(f,\sigma)}} &= \widehat{(g,\sigma')(1_B, f_!(\sigma))} \\
&= \widehat{(g, f_!(\sigma) \wedge \sigma')} \\
&= (1_C, g_!(f_!(\sigma) \wedge \sigma')) \\
&= (1_C, g_!(f_!(\sigma \wedge f^*(\sigma')))) \\
&= (1_C, (gf)_!(\sigma \wedge f^*(\sigma'))) \\
&= \widehat{(gf, \sigma \wedge f^*(\sigma'))} \\
&= \widehat{(g,\sigma')(f,\sigma)},
\end{aligned}$$

and therefore

$$\widehat{(g,\sigma')\widehat{(f,\sigma)}} = \widehat{(g,\sigma')(f,\sigma)}.$$

□

Examples

1. Suppose that $\mathbf{C}$ is a category. Then $\mathcal{S}_{rs}(1_{\mathbf{C}}) = \mathbf{C}$, which is a range restriction category with the trivial restriction and range structures.

2. For each range restriction category $\mathbf{C}$, $\mathcal{S}_{rs}(\partial_{\mathbf{C}})$ is the range restriction category with the same objects as $\mathbf{C}$ while a map from A to B in $\mathcal{S}_{rs}(\partial_{\mathbf{C}})$ is a pair (f, e)

with a map $f : A \to B$ in $\mathbf{C}$ and a restriction idempotent $e \leq \overline{f}$ over A in $\mathbf{C}$, the composition is given by $(g, e_B)(f, e_A) = (gf, e_A \wedge \overline{e_B f}) = (gf, \overline{e_B f} e_A)$ for any maps $(f, e_A) : A \to B$ and $(g, e_B) : B \to C$, and the range and restriction are given by $\overline{(f, e_A)} = (1_A, e_A)$ and $\widehat{(f, e_A)} = (1_A, \widehat{f e_A})$.

4.1.2 Category of Range Stable Meet Semilattice Fibrations and $\mathbf{rrCat}_0$

We begin with:

Category of Range Stable Meet Semilattice Fibrations

Let $\mathbf{rsFib}_0$ be the category with

objects: range stable meet semilattice fibrations: $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$;

maps: a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is a pair (F, F'), where $F : \mathbf{X} \to \mathbf{Y}$ and $F' : \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}}$ are functors such that

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \xrightarrow{F'} & \tilde{\mathbf{Y}} \\ {\scriptstyle \delta_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{Y}}} \\ \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \end{array}$$

commutes and for any map $f : A \to B$ in $\mathbf{X}$ and any $\sigma_A \in \delta_{\mathbf{X}}^{-1}(A), \sigma_B, \sigma'_B \in \delta_{\mathbf{X}}^{-1}(B)$, the following conditions are satisfied:

[sfM.1] $F'(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{Y}}^{-1}(F(A))}$,

[sfM.2] $F'(\sigma_B \wedge \sigma'_B) = F'(\sigma_B) \wedge F'(\sigma'_B)$,

[sfM.3] $F'(f^*(\sigma_B)) = (F(f))^*(F'(\sigma_B))$,

[rsfM.1] $F'(f_!(\sigma_A)) = (F(f))_!(F'(\sigma_A))$;

composition: for any maps $(F,F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ and $(G,G') : (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y}) \to (\delta_{\mathbf{Z}} : \tilde{\mathbf{Z}} \to \mathbf{Z})$, $(G,G')(F,F') = (GF, G'F')$;

identities: $1_{(\delta_{\mathbf{X}}:\tilde{\mathbf{X}}\to\mathbf{X})} = (1_{\mathbf{X}}, 1_{\tilde{\mathbf{X}}})$.

Functor $\mathcal{S}_{rs} : \mathbf{rsFib}_0 \to \mathbf{rrCat}_0$

By Proposition 4.1.4, each range stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ gives rise to a range restriction category $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$. Suppose now that $(F,F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is map in $\mathbf{rsFib}_0$. Then by Lemma 3.2.5, we have the restriction functor $\mathcal{S}_{rs}(F,F') : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathcal{S}_{rs}(\delta_{\mathbf{Y}})$ sending $(f,\sigma_A) : A \to B$ to $(F(f), F'(\sigma)) : F(A) \to F(B)$. Now we have:

Lemma 4.1.5 *If $(F,F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is map in $\mathbf{rsFib}_0$, then $\mathcal{S}_{rs}(F,F') : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathcal{S}_{rs}(\delta_{\mathbf{Y}})$, defined by taking $(f,\sigma) : A \to B$ in $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$ to $(F(f), F'(\sigma)) : F(A) \to F(B)$ in $\mathcal{S}_{rs}(\delta_{\mathbf{Y}})$, is a range restriction functor.*

PROOF: It remains to prove that $\mathcal{S}_{rs}(F,F') : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathcal{S}_{rs}(\delta_{\mathbf{Y}})$ is a range functor. For any map $(f,\sigma) : A \to B$ in $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$,

$$
\begin{aligned}
\mathcal{S}_{rs}(F,F')(\widehat{(f,\sigma)}) &= \mathcal{S}_{rs}(F,F')(1_B, f_!(\sigma)) \\
&= (F(1_B), F'(f_!(\sigma))) \\
&= (1_{F(B)}, (F(f))_!(F'(\sigma))) \text{ (by } [\mathbf{rsfM.1}]) \\
&= (\widehat{F(f), F'}(\sigma)) \\
&= \mathcal{S}_{rs}(\widehat{F,F'})(f,\sigma).
\end{aligned}
$$

Hence $\mathcal{S}_{rs}(F,F') : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathcal{S}_{rs}(\delta_{\mathbf{Y}})$ is a range restriction functor, as desired. □

It is easy to check that $\mathcal{S}_{rs} : \mathbf{rsFib}_0 \to \mathbf{rrCat}_0$ given by

$$\begin{array}{ccc} (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) & \mapsto & \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \\ {\scriptstyle (F,F')}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle \mathcal{S}_{rs}(F,F')} \\ (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y}) & \mapsto & \mathcal{S}_{rs}(\delta_{\mathbf{Y}}) \end{array}$$

is a functor. So we have:

Lemma 4.1.6 $\mathcal{S}_{rs} : \mathbf{rsFib}_0 \to \mathbf{rrCat}_0$, *sending* $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ *in* $\mathbf{rsFib}_0$ *to* $\mathcal{S}_{rs}(F, F') : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathcal{S}_{rs}(\delta_{\mathbf{Y}})$ *in* $\mathbf{rrCat}_0$, *is a functor.*

Let $\mathbf{Y}$ be a range restriction category, then by Lemma 4.1.2, $\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y}$ is range stable meet semilattice fibration. If $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ is a map in $\mathbf{rsFib}_0$, then by Lemma 3.2.7, there is a restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathbf{Y}$ sending $(f, \sigma) : A \to B$ to $(F(f))e_\sigma : F(A) \to F(B)$, where the restriction idempotent e_σ is determined by $F'(\sigma) = (F(A), e_\sigma) \in \mathbf{r}(\mathbf{Y})$. Similarly, we have:

Lemma 4.1.7 *Let* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *be a range stable meet semilattice fibration and let* $\mathbf{Y}$ *be a range restriction category. If* $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ *is a map in* $\mathbf{rsFib}_0$, *then there is a range restriction functor* $F^{\delta_{\mathbf{X}}} : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathbf{Y}$ *sending* $(f, \sigma) : A \to B$ *to* $(F(f))e_\sigma : F(A) \to F(B)$, *where the restriction idempotent* e_σ *is determined by* $F'(\sigma) = (F(A), e_\sigma) \in \mathbf{r}(\mathbf{Y})$.

PROOF: By Lemma 3.2.7, $F^{\delta_{\mathbf{X}}} : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathbf{Y}$ is a restriction functor. For any map

$(f,\sigma) : A \to B$ in $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$, since

$$\begin{aligned}
(F(B), e_{f_!(\sigma)}) &= F'(f_!(\sigma)) \\
&= (F(f))_!(F'(\sigma)) \text{ (by [}\mathbf{rsfM.1}\text{])} \\
&= (F(f))_!(F(A), e_\sigma) \\
&= (F(B), (\widehat{F(f)})e_\sigma),
\end{aligned}$$

we have $e_{f_!(\sigma)} = (\widehat{F(f)})e_\sigma$. Hence

$$\begin{aligned}
F^{\delta_{\mathbf{X}}}((\widehat{f,\sigma})) &= F^{\delta_{\mathbf{X}}}(1_B, f_!(\sigma)) \\
&= F(1_B)e_{f_!(\sigma)} \\
&= e_{f_!(\sigma)} \\
&= (\widehat{F(f)})e_\sigma \\
&= \widehat{F^{\delta_{\mathbf{X}}}(f,\sigma)},
\end{aligned}$$

and therefore $F^{\delta_{\mathbf{X}}}$ is a range restriction functor. □

Functor $\mathcal{R}_{rs} : \mathbf{rrCat}_0 \to \mathbf{rsFib}_0$

If $F : \mathbf{X} \to \mathbf{Y}$ is a range restriction functor, then we have a functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ given by

$$\begin{array}{ccc}
(A, e_A) & \mapsto & (F(A), F(e_A)) \\
{\scriptstyle f}\downarrow & \mapsto & \downarrow{\scriptstyle F(f)} \\
(B, e_B) & \mapsto & (F(B), F(e_B))
\end{array}$$

By Lemma 3.2.9, there is a unique functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ such that

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$. But it is a map in $\mathbf{rsFib}_0$ if $F : \mathbf{X} \to \mathbf{Y}$ is a range restriction functor:

Lemma 4.1.8 *If $F : \mathbf{X} \to \mathbf{Y}$ is a range restriction functor, then there is a unique functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ such that*

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{rsFib}_0$.

PROOF: Clearly, it suffices to prove that $(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ satisfies the condition [**rsfM.1**]. For any map $f : A \to B$ in $\mathbf{X}$ and $(A, e) \in \partial_{\mathbf{X}}^{-1}(A)$,

$$\begin{aligned}
\mathbf{r}(F)(f_!(A,e)) &= \mathbf{r}(F)(B, \widehat{fe}) \\
&= (F(B), F(\widehat{fe})) \\
&= (F(B), \widehat{F(f)F(e)}) \\
&= (F(f))_!(F(A), F(e)) \\
&= (F(f))_!(\mathbf{r}(F)(A,e)),
\end{aligned}$$

as desired. $\square$

Lemma 4.1.9 $\mathcal{R}_{rs} : \mathbf{rrCat}_0 \to \mathbf{rsFib}_0$*, given by taking* $F : \mathbf{X} \to \mathbf{Y}$ *in* $\mathbf{rrCat}_0$ *to* $(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ *in* $\mathbf{rsFib}_0$*, is a functor.*

PROOF: For any range restriction functors $F : \mathbf{X} \to \mathbf{Y}$ and $G : \mathbf{Y} \to \mathbf{Z}$, we have

$$\mathcal{R}_{rs}(GF) = (GF, \mathbf{r}(GF)) = (GF, \mathbf{r}(G)\mathbf{r}(F)) = (G, \mathbf{r}(G))(F, \mathbf{r}(F)) = \mathcal{R}_{rs}(G)\mathcal{R}_{rs}(F).$$

Clearly, $\mathcal{R}_{rs}(1_{\mathbf{X}}) = (1_{\mathbf{X}}, 1_{\mathbf{r}(\mathbf{X})})$. Hence $\mathcal{R}_{rs}$ is a functor. □

Adjunction $\mathcal{S}_{rs} \dashv \mathcal{R}_{rs}$

For a given range stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, we can form a functor $I_{\mathbf{X}} : \mathbf{X} \to \mathcal{S}_{rs}(\delta_{\mathbf{X}})$ by sending $f : A \to B$ to $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$. Also, we have a functor

$$I_{\mathbf{X}}^{\delta_{\mathbf{X}}} : \tilde{\mathbf{X}} \to \mathbf{r}(\mathcal{S}_{rs}(\delta_{\mathbf{X}}))$$

given by

$$\begin{array}{ccc} U & \mapsto & (\delta_{\mathbf{X}}(U), (1_{\delta_{\mathbf{X}}(U)}, U)) \\ {\scriptstyle f}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle (\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))}))} \\ V & \mapsto & (\delta_{\mathbf{X}}(V), (1_{\delta_{\mathbf{X}}(V)}, V)) \end{array}$$

Lemma 4.1.10 *If* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *is a range stable meet semilattice fibration, then* $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ *is a map from* $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ *to* $(\partial_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_{rs}(\delta_{\mathbf{X}})) \to \mathcal{S}_{rs}(\delta_{\mathbf{X}}))$ *in* $\mathbf{rsFib}_0$.

PROOF: By Lemma 3.2.11, we have $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ is map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\partial_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})}$:

$\mathbf{r}(\mathcal{S}_{rs}(\delta_{\mathbf{X}})) \to \mathcal{S}_{rs}(\delta_{\mathbf{X}}))$ in $\mathbf{sFib}_0$. For any map $f : A \to B$ in $\mathbf{X}$ and $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$,

$$\begin{aligned}
I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(f_!(\sigma)) &= (\delta_{\mathbf{X}}(f_!(\sigma)), (1_{\delta_{\mathbf{X}}(f_!(\sigma))}, f_!(\sigma))) \\
&= (B, (1_B, f_!(\sigma))) \\
&= (B, \widehat{(f, \sigma)}) \\
&= (B, \widehat{(f, \sigma \wedge f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})}) \\
&= (B, \widehat{(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))}(1_A, \sigma)) \\
&= (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))_!(A, (1_A, \sigma)) \\
&= (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))_!(\delta_{\mathbf{X}}(\sigma), (1_{\delta_{\mathbf{X}}(\sigma)}, \sigma)) \\
&= (I_{\mathbf{X}}(f))_!(I_{\mathbf{X}}^{\delta_{\mathbf{X}}}(\sigma)).
\end{aligned}$$

Hence, condition [**rsfM.1**] holds true, and therefore $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ is map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\partial_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_{rs}(\delta_{\mathbf{X}})) \to \mathcal{S}_{rs}(\delta_{\mathbf{X}}))$ in $\mathbf{rsFib}_0$. □

For any range stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, by Lemma 4.1.10, there exists a map $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}_{rs}(\delta_{\mathbf{X}})) \to \mathcal{S}_{rs}(\delta_{\mathbf{X}}))$ in $\mathbf{rsFib}_0$. This map turns out to be the unit of the adjunction $\mathcal{S}_{rs} \dashv \mathcal{R}_{rs}$. In fact, let $\mathbf{Y}$ be a range restriction category and $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to \mathcal{R}_{rs}(\mathbf{Y})$ any map in $\mathbf{rsFib}_0$. By Lemma 4.1.7, there is a range restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathbf{Y}$. It is easy to check that

$$(F^{\delta_{\mathbf{X}}}, \mathbf{r}(F^{\delta_{\mathbf{X}}}))(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) = (F, F').$$

If $G : \mathcal{S}_{rs}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to \mathbf{Y}$ is range restriction functor such that

$$(G, \mathbf{r}(G))(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}}) = (F, F'),$$

then $GI_{\mathbf{X}} = F$ and $\mathbf{r}(G)I_{\mathbf{X}}^{\delta_{\mathbf{X}}} = F'$. Hence, for any map $f : A \to B$ in $\mathbf{X}$, G must map A to $F(A)$ and must map $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$ to $F(f) : F(A) \to F(B)$. Since $e_{f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})} = \overline{F(f)}$, $G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = F(f) = F^{\delta_{\mathbf{X}}}(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$. For any map $(f, \sigma) : A \to B$ in $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$,

$$\begin{aligned}
\mathbf{r}(G)(I_{\mathbf{X}}^{\delta_{\mathbf{X}}})(\sigma) &= \mathbf{r}(G)(A, (1_A, \sigma)) \\
&= (G(A), G(1_A, \sigma)) \\
&= F'(\sigma) \\
&= (F(A), e_\sigma),
\end{aligned}$$

and so $G(1_A, \sigma) = e_\sigma$. Since $(f, \sigma) = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))(1_A, \sigma)$ and G is a range restriction functor,

$$\begin{aligned}
G(f, \sigma) &= G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))G(1_A, \sigma) \\
&= F(f)e_\sigma \\
&= F^{\delta_{\mathbf{X}}}(f, \sigma).
\end{aligned}$$

Then $G = F^{\delta_{\mathbf{X}}}$ and so the uniqueness of $F^{\delta_{\mathbf{X}}}$ follows. Therefore, there is a unique range restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}_{rs}(\delta_{\mathbf{X}}) \to \mathbf{Y}$ such that

$$\begin{array}{ccc} (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) & \xrightarrow{(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})} & \mathcal{R}_{rs}(\mathcal{S}_{rs}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})) \\ & \searrow^{(F,F')} & \downarrow{\scriptstyle \mathcal{R}_{rs}(F^{\delta_{\mathbf{X}}})} \\ & & \mathcal{R}_{rs}(\mathbf{Y}) \end{array} \qquad \begin{array}{c} \mathcal{S}_{rs}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \\ \exists! \downarrow F^{\delta_{\mathbf{X}}} \\ \mathbf{Y} \end{array}$$

commutes. Hence $\mathcal{S}_{rs} \dashv \mathcal{R}_{rs}$. The counit $\varepsilon : \mathcal{S}_{rs}\mathcal{R}_{rs} \to 1_{\mathbf{rrCat}_0}$ of $\mathcal{S}_{rs} \dashv \mathcal{R}_{rs}$ is given by $\varepsilon_{\mathbf{C}} : \mathcal{S}_{rs}(\mathcal{R}_{rs}(\mathbf{C})) \to \mathbf{C}$ which sends $(f, e_A) : A \to B$ to $fe_A : A \to B$, where e_A is a restriction idempotent over A such that $e_A \leq \overline{f}$, for each range restriction category $\mathbf{C}$. Since each $\varepsilon_{\mathbf{C}}$ is a split epic in $\mathbf{Cat}_0$, $\varepsilon_{\mathbf{C}}$ is an epic in $\mathbf{rCat}_0$. Hence $\mathcal{R}_{rs}$ is faithful. We proved:

Theorem 4.1.11 *There is an adjunction:*

$$\mathbf{rrCat}_0 \underset{\mathcal{R}_{rs}}{\overset{\mathcal{S}_{rs}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{rsFib}_0$$

with a faithful functor $\mathcal{R}_{rs}$.

4.1.3 The Image of $\mathcal{S}_{rs}$: Fibered Range Restriction Categories

In Subsection 3.2.4, we characterized the image of the functor $\mathcal{S}_s$. In this subsection, we shall specify the class of range restriction categories, which is the image of $\mathcal{S}_{rs}$: $\mathbf{rsFib}_0 \to \mathbf{rrCat}_0$.

For a pair of objects A, B in a range restriction category $\mathbf{C}$, we define

$$\mathrm{map}_{\mathbf{C}}^{\max}(A, B) = \{f \in \mathrm{map}_{\mathbf{C}}(A, B) \mid f \leq h \ \text{ implies } \ h = f \ \text{ in } \ \mathrm{map}_{\mathbf{C}}(A, B)\},$$

where the order $\leq$ is defined in Lemma 3.2.13. A range restriction category $\mathbf{C}$ is called *a fibered range restriction category* if it satisfies the following two conditions:

[**M.1**] For any objects A, B and any $f \in \mathrm{map}_{\mathbf{C}}(A, B)$, there is a unique $m_f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$ such that $f \leq m_f$;

[**M.2**] For any objects A, B, C, $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$, and $g \in \mathrm{map}^{\max}_{\mathbf{C}}(B, C)$, $gf \in \mathrm{map}^{\max}_{\mathbf{C}}(A, C)$.

Clearly, for any map $f : A \to B$ in a range restriction category $\mathbf{C}$, $m_{\overline{f}} = m_{\widehat{f}} = 1_A$ and $1_A \in \mathrm{map}^{\max}_{\mathbf{C}}(A, A)$.

Since a range stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a stable meet semilattice fibration, and $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$ is the range restriction category with the same objects, the same maps, and the same restriction as $\mathcal{S}_s(\delta_{\mathbf{X}})$ and with the range given by $\widehat{(f, \sigma)} = (1, f_!(\sigma))$, the range restriction category $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$ satisfies the conditions [**M.1**] and [**M.2**]. In fact, for any objects A, B,

$$m_{(f,\sigma)} = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})),$$

$$\mathrm{map}^{\max}_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})}(A, B) = \{(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \mid f \in \mathrm{map}_{\mathbf{X}}(A, B)\},$$

and for any $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \in \mathrm{map}^{\max}_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})}(A, B), (g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \in \mathrm{map}^{\max}_{\mathcal{S}_{rs}(\delta_{\mathbf{X}})}(B, C)$,

$$(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}))(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = (gf, (gf)^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})).$$

Hence $\mathcal{S}_{rs}(\delta_{\mathbf{X}})$ is a fibered range restriction category.

Let $\mathbf{C}$ be a fibered range restriction category. Define $\mathbf{C}_{\max}$ by following data:

objects: the same as the objects of $\mathbf{C}$;

maps: for any objects A, B, $\mathrm{map}_{\mathbf{C}_{\max}}(A, B) = \mathrm{map}_{\mathbf{C}}^{\max}(A, B)$;

composition: the same as in $\mathbf{C}$.

Then, by [**M.2**], $\mathbf{C}_{\max}$ is a category. We define $\tilde{\mathbf{C}}_{\max}$ to be the category given by

objects: (A, e_A), where e_A is a restriction idempotent over A in $\mathbf{C}$;

maps: a map f from (A, e_A) to (B, e_B) is a map $f \in \mathrm{map}_{\mathbf{C}}^{\max}(A, B)$ such that $e_A = \overline{e_B f} e_A$;

composition: the same as in $\mathbf{C}$.

Obviously, there is a forgetful functor $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$, which forgets restriction idempotents.

Lemma 4.1.12 *For any fibered range restriction category* $\mathbf{C}$*, the forgetful functor* $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ *is a range stable meet semilattice fibration.*

PROOF: By Lemma 3.2.14, $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ is a stable meet semilattice fibration. For any map $f : A \to B$ in $\mathbf{C}_{\max}$ and any $(A, e_A) \in \partial^{-1}_{\mathbf{C}_{\max}}(A)$, we define $f_! : \partial^{-1}_{\mathbf{C}_{\max}}(A) \to \partial^{-1}_{\mathbf{C}_{\max}}(B)$ by sending (A, e_A) to $(B, \widehat{f e_A})$. By the proof of Lemma 4.1.2, $f_!$ is a monotone map such that $(gf)_! = g_! f_!$ and the conditions [**rsF.1**], [**rsF.2**], and [**rsF.3**] are satisfied. Hence $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ is a range stable meet semilattice fibration. □

Proposition 4.1.13 *For any fibered range restriction category* $\mathbf{C}$*,* $\mathcal{S}_{rs}(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$.

PROOF: Define the functor $E : \mathcal{S}_{rs}(\partial_{\mathbf{C}_{\max}}) \to \mathbf{C}$ by sending $(f, (A, e_A)) : A \to B$ to $fe_A : A \to B$. Then it is a restriction functor. Note that

$$E(\widehat{f, (A, e_A)}) = \widehat{fe_A} = E(1_B, (B, \widehat{fe_A})) = E((f, \widehat{(A, e_A)})).$$

Hence E is a range restriction functor.

For any map $f : A \to B$ in $\mathbf{C}$, by [**M.1**], we can define $F : \mathbf{C} \to \mathcal{S}_s(\partial_{\mathbf{C}_{\max}})$ by sending $f : A \to B$ to $(m_f, (A, \overline{f})) : A \to B$. Then, by the proof of Proposition 3.2.15, F is a restriction functor. Note that

$$\begin{aligned} F(\widehat{f}) &= (m_{\widehat{f}}, (B, \overline{\widehat{f}})) \\ &= (1_B, (B, \widehat{m_f \overline{f}})) \\ &= (1_B, (m_f)_!(A, \overline{f})) \\ &= (\widehat{m_f, (A, \overline{f})}) \\ &= \widehat{F(f)}. \end{aligned}$$

Hence F is a range restriction functor. Now it is routine to check that $EF = 1_{\mathbf{C}}$ and $FE = 1_{\mathcal{S}_{rs}(\partial_{\mathbf{C}_{\max}})}$. Thus, $\mathcal{S}_{rs}(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$. □

4.2 Range Restriction Fibrations and Range Restriction Categories

We already saw that the category of restriction categories is equivalent to the category of restriction fibrations (Theorem 3.3.11). It is natural to ask whether range restriction categories can be characterized by some special fibrations. The purpose of this section is to answer this question by introducing the notion of range restriction fibrations and showing that the category of range restriction categories is equivalent to the category of range restriction fibrations.

4.2.1 Definition of Range Restriction Fibrations

Definition 4.2.1 *A fibration $\partial : \mathbf{D} \to \mathbf{C}$ is called* a range restriction fibration *if for each object X of $\mathbf{C}$, the fiber $\partial^{-1}(X)$ is a meet semilattice in which $E_1 \leq E_2$ if and only if there is a map from E_1 to E_2, and for any object E of $\partial^{-1}(X)$, there is a map $\varepsilon_E : X \to X$, and for each map $f : X \to Y$, there is a $\omega_f \in \partial^{-1}(Y)$ such that:*

[rF.1] $\varepsilon_{\top_{\partial^{-1}(X)}} = 1_X$,

[rF.2] $\varepsilon_E^*(\top_{\partial^{-1}(X)}) = E$, *where $\vartheta_{\varepsilon_E} : \varepsilon_E^*(\top_{\partial^{-1}(X)}) \to \top_{\partial^{-1}(X)}$ is the cartesian lifting of ε_E at $\top_{\partial^{-1}(X)}$,*

[rF.3] $\varepsilon_E \varepsilon_{E'} = \varepsilon_{E \wedge E'}$,

[rF.4] $\varepsilon_F(f) = f \varepsilon_{f^*(F)}$,

[rrF.1] $\varepsilon_{\varepsilon^*_{\omega_f}(\top_{\partial^{-1}(Y)})} = \varepsilon_{f^*(\top_{\partial^{-1}(Y)})}$,

[rrF.2] $\varepsilon_{\omega_f} f = f$,

[rrF.3] $\varepsilon_{\omega_{\varepsilon_{g^*(\top_{\partial^{-1}(Z)})}f}} = \varepsilon_{g^*(\top_{\partial^{-1}(Z)})}\varepsilon_{\omega_f}$,

[rrF.4] $\varepsilon_{\omega_{g\varepsilon_{\omega_f}}} = \varepsilon_{\omega_{gf}}$,

for any map $g : Y \to Z$ *in* $\mathbf{C}$ *and any* $E, E' \in \partial^{-1}(X)$ *and any* $F \in \partial^{-1}(Y)$.

One may easily give the definition of range restriction indexed categories by translating that of range restriction fibrations.

For example, for any category $\mathbf{C}$, the identity fibration $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is a range restriction fibration and is called the *trivial range restriction fibration over* $\mathbf{C}$.

If $\mathbf{C}$ is a range restriction category, then the forgetful functor $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a range restriction fibration, as show in the following lemma.

Lemma 4.2.2 *Let* $\mathbf{C}$ *be a range restriction category. Then the forgetful functor* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *is a range restriction fibration. We denote this range restriction fibration by* $\mathcal{R}_{rr}(\mathbf{C})$.

PROOF: By Lemma 3.3.2, $\partial_{\mathbf{C}}$ is a restriction fibration. To prove that it is a range restriction fibration, for any map $f : X \to Y$ in $\mathbf{C}$, we let $\omega_f = (Y, \widehat{f}) \in \partial_{\mathbf{C}}^{-1}(Y)$. It suffices to show that $\partial_{\mathbf{C}}$ satisfies range fibration axioms [**rrF**.1], [**rrF**.2], [**rrF**.3], and [**rrF**.4]. For any $f : X \to Y$, we have:

[rrF.1] $\varepsilon_{\varepsilon_f^*(\top_{\partial(Y)})} = \varepsilon_{\varepsilon_{\omega_f}^*(Y,1_Y)} = \varepsilon_{\widehat{f}^*(Y,1_Y)} = \varepsilon_{(Y,\overline{\widehat{f}})} = \varepsilon_{(Y,\widehat{f})} = \varepsilon_{\omega_f}$.

[rrF.2] $\varepsilon_{\omega_f} f = \varepsilon_{(Y,\widehat{f})} f = \widehat{f} f = f$.

[rrF.3]

$$\begin{aligned}
\varepsilon_{\omega_{\varepsilon_{g^*(\top_{\partial^{-1}(Z)})f}}} &= \varepsilon_{\omega_{\varepsilon_{g^*(Z,1_Z)f}}} \\
&= \varepsilon_{\omega_{\varepsilon_{(Y,\overline{g})f}}} \\
&= \varepsilon_{\omega_{\overline{g}f}} \\
&= \varepsilon_{(Y,\widehat{\overline{g}f})} \\
&= \widehat{\overline{g}f} \\
&= \overline{g}\widehat{f} \\
&= \varepsilon_{(Y,\overline{g})}\varepsilon_{(Y,\widehat{f})} \\
&= \varepsilon_{g^*(\top_{\partial^{-1}(Z)})}\varepsilon_{\omega_f}.
\end{aligned}$$

[rrF.4] $\varepsilon_{\omega_{g\varepsilon_{\omega_f}}} = \varepsilon_{\omega_g\varepsilon_{(Y,\widehat{f})}} = \varepsilon_{\omega_{g\widehat{f}}} = \varepsilon_{(Z,\widehat{g\widehat{f}})} = \widehat{g\widehat{f}} = \widehat{gf} = \varepsilon_{\omega_{gf}}$.

Therefore, $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a range restriction fibration, as desired. □

Similar to restriction fibrations, we shall see that all range restriction fibrations are of the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ (see Proposition 4.2.8 below).

4.2.2 Characterizations of Range Restriction Categories in Terms of Fibrations

Range restriction categories can be characterized by range restriction fibrations as shown in the following theorem.

Theorem 4.2.3 *A category* $\mathbf{C}$ *is a range restriction category if and only if there is a range restriction fibration* $\delta : \mathbf{D} \to \mathbf{C}$ *for some category* $\mathbf{D}$.

PROOF: If $\mathbf{C}$ is a range restriction category, then, by Lemma 4.2.2, there is a range restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$. Conversely, if there is a range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, then $\mathbf{C}$ is a restriction category with the restriction given by $\overline{f} = \varepsilon_{f^*(\top_{\partial^{-1}(Y)})}$ for each map $f : X \to Y$ in $\mathbf{C}$ because of the conditions [**rF**.**1**], [**rF**.**2**], [**rF**.**3**], and [**rF**.**4**]. On the other hand, if we define $\widehat{f}$ by $\widehat{f} = \varepsilon_{\omega_f}$, then $\mathbf{C}$ becomes a range restriction category, since the range fibration conditions [**rrF**.**1**], [**rrF**.**2**], [**rrF**.**3**], and [**rrF**.**4**] correspond to the four range axioms [**RR**.**1**], [**RR**.**2**], [**RR**.**3**], and [**RR**.**4**]. Hence, $\mathbf{C}$ is indeed a range restriction category. □

4.2.3 The Category of Range Restriction Fibrations is Equivalent to $\mathbf{rrCat}_0$

The objective of this subsection is to prove that the category of range restriction fibrations is equivalent to $\mathbf{rrCat}_0$. To do so, we use the techniques developed in Section 3.3.3 leading to the fact that $\mathbf{rFib}_0 \approx \mathbf{rCat}_0$, but here, we have to worry about the range structures.

The Category of Range Restriction Fibrations and Functors $\mathcal{R}_{rr}$ and $\mathcal{E}_{rr}$

Let $\mathbf{rrFib}_0$ be the category with

objects: range restriction fibrations: $\delta : \mathbf{D} \to \mathbf{C}$;

maps: a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\delta' : \mathbf{D}' \to \mathbf{C}')$ is a pair (F, F'), where $F : \mathbf{C} \to \mathbf{C}'$ and $F' : \mathbf{D} \to \mathbf{D}'$ are functors such that

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F'} & \mathbf{D}' \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta'} \\ \mathbf{C} & \xrightarrow{F} & \mathbf{C}' \end{array}$$

commutes and for any map $f : X \to Y$ in $\mathbf{C}$ and any $E \in \delta^{-1}(X), W \in \delta^{-1}(Y)$, the following conditions are satisfied:

[pR.1] $F'(\top_{\delta^{-1}(X)}) = \top_{(\delta')^{-1}(F(X))}$,

[pR.2] $F(\varepsilon_E) = \varepsilon_{F'(E)}$,

[pR.3] $F'(f^*(W)) = (F(f))^*(F'(W))$,

[pRR.1] $F'(w_f) = w_{F(f)}$;

composition and **identities** are defined by: $(F_2, F_2')(F_1, F_1') = (F_2F_1, F_2'F_1')$, and $1_{(\delta:\mathbf{D}\to\mathbf{C})} = (1_{\mathbf{C}}, 1_{\mathbf{D}})$.

Clearly, $\mathbf{rrFib}_0$ is a subcategory of $\mathbf{rFib}_0$. By Lemma 4.2.2, each range restriction category $\mathbf{C}$ gives rise to a range restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$, denoted by $\mathcal{R}_{rr}(\mathbf{C})$. If $F : \mathbf{D} \to \mathbf{C}$ is a range restriction functor, then we have a range restriction functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{D}) \to \mathbf{r}(\mathbf{C})$ defined by

$$\begin{array}{ccc} (D_1, e_1) & \mapsto & (F(D_1), F(e_1)) \\ {\scriptstyle d}\downarrow & \mapsto & \downarrow{\scriptstyle F(d)} \\ (D_2, e_2) & \mapsto & (F(D_2), F(e_2)) \end{array}$$

such that

$$\begin{array}{ccc} \mathbf{r}(\mathbf{D}) & \xrightarrow{\mathbf{r}(F)} & \mathbf{r}(\mathbf{C}) \\ {\scriptstyle \partial_{\mathbf{D}}}\downarrow & & \downarrow{\scriptstyle \partial_{\mathbf{C}}} \\ \mathbf{D} & \xrightarrow{F} & \mathbf{C} \end{array}$$

commutes. Obviously, $(\mathbf{r}(F), F)$ satisfies [**pR**.**1**], [**pR**.**2**], and [**pR**.**3**]. For any map

$f : D_1 \to D_2$ in $\mathbf{D}$, since

$$\mathbf{r}(F)(w_f) = \mathbf{r}(F)(D_2, \widehat{f}) = (F(D_2), F(\widehat{f})) = (F(D_2), \widehat{F(f)}) = w_{F(f)},$$

[**pRR.1**] is also satisfied. Hence $\mathcal{R}_{rr}(F) = (F, \mathbf{r}(F)) : \mathcal{R}_{rr}(\mathbf{D}) \to \mathcal{R}_{rr}(\mathbf{C})$ is actually a map in $\mathbf{rrFib}_0$. Therefore,

Lemma 4.2.4 *$\mathcal{R}_{rr} : \mathbf{rrCat}_0 \to \mathbf{rrFib}_0$, taking $F : \mathbf{D} \to \mathbf{C}$ to $\mathcal{R}_{rr}(F) : (\partial_{\mathbf{D}} : \mathbf{r}(\mathbf{D}) \to \mathbf{D}) \to (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$, is a functor.*

Lemma 4.2.5 *$\mathcal{E}_{rr} : \mathbf{rrFib}_0 \to \mathbf{rrCat}_0$, sending $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ to $F : \mathbf{C} \to \mathbf{C}'$, is a functor.*

PROOF: For any map $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ in $\mathbf{rrFib}_0$, (F, F') satisfies [**pR.1**], [**pR.2**], [**pR.3**], and [**pRR.1**]. Then

$$\begin{aligned} F(\overline{f}) &= F(\varepsilon_{f^*(\top_{\delta^{-1}(Y)})}) \\ &= \varepsilon_{F'(f^*(\top_{\delta^{-1}(Y)}))} \\ &= \varepsilon_{(F(f))^*(F'(\top_{\delta^{-1}(Y)}))} \\ &= \varepsilon_{(F(f))^*(\top_{(\delta')^{-1}(F(Y))})} \\ &= \overline{F(f)} \end{aligned}$$

and

$$F(\widehat{f}) = F(\varepsilon_{w_f}) = \varepsilon_{F'(w_f)} = \varepsilon_{w_{F(f)}} = \widehat{F(f)}.$$

Hence $F : \mathbf{C} \to \mathbf{C}'$ is a range restriction functor and therefore $\mathcal{E}_{rr}$ is well-defined. It is routine to check that $\mathcal{E}_{rr} : \mathbf{rrFib}_0 \to \mathbf{rrCat}_0$ is a functor. $\square$

Equivalence Adjunction $\mathcal{E}_{rr} \dashv \mathcal{R}_{rr}$

Given any range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, $\mathbf{C}$ is a range restriction category with the restriction and range structures induced by δ. Assume that $F : \mathbf{C} \to \mathbf{E}$ is a range restriction functor. As in Section 3.3.3, we can construct a functor $F_\delta : \mathbf{D} \to \mathbf{r}(\mathbf{E})$ by

$$\begin{array}{ccc} D_1 & \mapsto & ((F\delta)(D_1), F(\varepsilon_{D_1})) \\ {\scriptstyle f}\downarrow & \mapsto & \downarrow{\scriptstyle (F\delta)(f)} \\ D_2 & \mapsto & ((F\delta)(D_2), F(\varepsilon_{D_2})) \end{array}$$

such that

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F_\delta} & \mathbf{r}(\mathbf{E}) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \partial_\mathbf{E}} \\ \mathbf{C} & \xrightarrow{F} & \mathbf{E} \end{array}$$

commutes. Certainly, (F, F_δ) satisfies [**pR.1**], [**pR.2**], and [**pR.3**]. For any map $f : X \to Y$ in $\mathbf{C}$, note that

$$F_\delta(w_f) = ((F\delta)(w_f), F(\varepsilon_{w_f})) = (F(Y), F(\widehat{f})) = (F(Y), \widehat{F(f)}) = w_{F(f)}.$$

Hence [**pRR.1**] is also satisfied. Therefore (F, F_δ) is a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\partial_\mathbf{E} : \mathbf{r}(\mathbf{E}) \to \mathbf{E})$ in $\mathbf{rrFib}_0$. By the same process we used in Lemma 3.3.7, we have:

Lemma 4.2.6 *Let $\delta : \mathbf{D} \to \mathbf{C}$ be a range restriction fibration and $\mathbf{C}$ the range restriction category with the range and restriction structures induced by δ. Then (F, G) is a map from $(\delta : \mathbf{D} \to \mathbf{C})$ to $(\partial_\mathbf{E} : \mathbf{r}(\mathbf{E}) \to \mathbf{E})$ in $\mathbf{rrFib}_0$ if and only if*

$G = F_\delta$. In particular, if $F : \mathbf{C} \to \mathbf{D}$ is a range restriction functor, then (F, G) is a map from $(\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ to $(\partial_{\mathbf{D}} : \mathbf{r}(\mathbf{D}) \to \mathbf{D})$ in $\mathbf{rrFib}_0$ if and only if $G = \mathbf{r}(F)$.

For any range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, by Lemma 4.2.6, there exists a map $(1_{\mathbf{C}}, 1_\delta) : (\delta : \mathbf{D} \to \mathbf{C}) \to (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ taking $f : D_1 \to D_2$ to $\delta(f) : (\delta(D_1), \varepsilon_{D_1}) \to (\delta(D_2), \varepsilon_{D_2})$ in $\mathbf{rrFib}_0$. As in Lemma 3.3.8, this map turns out to be the unit of the adjunction $\mathcal{E}_{rr} \dashv \mathcal{R}_{rr}$. Clearly, $\mathcal{E}_{rr}\mathcal{R}_{rr} = 1_{\mathbf{rrCat}_0}$. We have:

Lemma 4.2.7 *There is an adjunction*

$$\mathbf{rrCat}_0 \underset{\mathcal{R}_{rr}}{\overset{\mathcal{E}_{rr}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{rrFib}_0$$

with identity counit so that $\mathcal{R}_{rr}$ is full and faithful.

Also, all range restriction fibrations are of the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$.

Proposition 4.2.8 *Each range restriction fibration can be written in the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ for some range restriction category $\mathbf{C}$.*

Proof: By Theorem 4.2.3, each range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$ gives rise to range restriction category $\mathbf{C}$, and so we have a restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$. By Lemma 4.2.6, there is a map $(1_{\mathbf{C}}, 1_\delta) : (\delta : \mathbf{D} \to \mathbf{C}) \to \mathcal{R}_{rr}(\mathbf{C})$. If $f : (C_1, \varepsilon_{E_1}) \to (C_2, \varepsilon_{E_2})$ is a map in $\mathbf{r}(\mathbf{C})$, then $\varepsilon_{E_1} \leq \varepsilon_{f^*E_2}$, and so $E_1 \leq f^*E_2$ by Lemma 3.3.3. Hence, there is a unique map leq $: E_1 \to f^*E_2$ in $\delta^{-1}(C_1)$, and so a map $\text{lift}(f) = \vartheta_f \text{leq} : E_1 \to E_2$ satisfying $\delta(\text{lift}(f)) = f$, where $\vartheta_{f,E_2} : f^*E_2 \to E_2$ is the cartesian lifting of f at E_2. As in Proposition 3.3.10, we can define a functor $G : \mathbf{r}(\mathbf{C}) \to \mathbf{D}$ by sending $f : (C_1, \varepsilon_{E_1}) \to (C_2, \varepsilon_{E_2})$ to $\text{lift}(f) : E_1 \to E_2$. Of course, $(1_{\mathbf{C}}, G)$ satisfies [**pR.1**], [**pR.2**], and [**pR.3**]. For any map $f : C_1 \to C_2$ in $\mathbf{C}$, $G(w_f) = G(C_2, \varepsilon_{w_f}) = w_{1_{\mathbf{C}}(f)}$. Hence [**pRR.1**] is also satisfied and therefore $(1_{\mathbf{C}}, G)$ is a map from $\mathcal{R}_{rr}(\mathbf{C})$ to $(\delta : \mathbf{D} \to \mathbf{C})$ in $\mathbf{rrFib}_0$. Obviously, $(1_{\mathbf{C}}, G)(1_{\mathbf{C}}, 1_\delta) = 1_{(\delta:\mathbf{D}\to\mathbf{C})}$ and $(1_{\mathbf{C}}, 1_\delta)(1_{\mathbf{C}}, G) = 1_{\mathcal{R}_{rr}(\mathbf{C})}$. Hence $\mathcal{R}_{rr}(\mathbf{C}) \cong (\delta : \mathbf{D} \to \mathbf{C})$. □

Combining Lemma 4.2.7 and Proposition 4.2.8, $\mathcal{R}_{rr}$ is not only full and faithful but also surjective on objects. Thus, we have:

Theorem 4.2.9 *There is an adjoint equivalence*

$$\mathbf{rrCat}_0 \underset{\mathcal{R}_{rr}}{\overset{\mathcal{E}_{rr}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{rrFib}_0$$

with identity counit.

4.2.4 Inverse Image Functors and Direct Image Functors

Given a range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$, for each map $f : X \to Y$, we have the inverse image functor $f^* : \delta^{-1}(Y) \to \delta^{-1}(X)$. The objective of this section is to answer which f^* have left adjoints. Since δ is of the form $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ for some range restriction category $\mathbf{C}$ by Proposition 4.2.8, we only need to consider the range restriction fibration $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ for some range restriction category $\mathbf{C}$. We need the following lemma.

Lemma 4.2.10 *In a range restriction category, if* $\mathrm{dom}(g) = \mathrm{codom}(f)$ *then*

$$\widehat{f} \leq \overline{g} \Leftrightarrow \overline{f} \leq \overline{gf}.$$

PROOF: If $\widehat{f} \leq \overline{g}$, then $\widehat{f} = \overline{g}\widehat{f}$. Hence

$$\begin{aligned}
\overline{gf}\,\overline{f} &= \overline{gf} \text{ (since } \overline{f}\,\overline{gf} = \overline{gf}) \\
&= \overline{\overline{g}f} \\
&= \overline{\overline{g}\widehat{f}f} \text{ (by [}\mathbf{RR.2}\text{])} \\
&= \overline{\widehat{f}f} \\
&= \overline{f},
\end{aligned}$$

and therefore $\overline{f} \leq \overline{gf}$. Conversely, if $\overline{f} \leq \overline{gf}$, then $\overline{f} = \overline{f}\,\overline{gf} = \overline{gf}$. Hence

$$\begin{aligned}
\overline{g}\widehat{f} &= \widehat{\overline{g}f} \text{ (by [}\mathbf{RR.3}\text{])} \\
&= \widehat{f\overline{gf}} \text{ (by [}\mathbf{R.4}\text{])} \\
&= \widehat{f\overline{f}} \\
&= \widehat{f} \text{ (by [}\mathbf{R.1}\text{])},
\end{aligned}$$

and therefore $\widehat{f} \leq \overline{g}$. □

Assume that $\mathbf{C}$ is a range restriction category. Since $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a fibration, for each map $f : X \to Y$ in $\mathbf{C}$, one has the inverse image functor $f^* : \partial_{\mathbf{C}}^{-1}(Y) \to \partial_{\mathbf{C}}^{-1}(X)$ taking $(Y, e_Y) \leq (Y, e'_Y)$ to $(X, \overline{e_Y f}) \leq (X, \overline{e'_Y f})$. On the other hand, we define the direct image functor $f_! : \partial_{\mathbf{C}}^{-1}(X) \to \partial_{\mathbf{C}}^{-1}(Y)$ by sending $(X, e_X) \leq (X, e'_X)$ to $(Y, \widehat{fe_X}) \leq (Y, \widehat{fe'_X})$. If $(X, e_X) \leq (X, e'_X)$ in the fiber $\partial_{\mathbf{C}}^{-1}(X)$, then $e_X = e'_X e_X =$

$e_X e'_X$ and so

$$\begin{aligned}\widehat{fe'_X}\widehat{fe_X} &= \widehat{fe'_X e_X}\widehat{fe'_X} \\ &= \widehat{fe'_X e_X} \text{ (by [}\mathbf{R.4}\text{])} \\ &= \widehat{fe_X}.\end{aligned}$$

Hence, $f_! 1_X = 1_Y : (Y, \widehat{fe_X}) \to (Y, \widehat{fe'_X})$ is a map in $\partial_{\mathbf{C}}^{-1}(Y)$ whenever $1_X : (X, e_X) \to (X, e'_X)$ is a map in $\partial_{\mathbf{C}}^{-1}(X)$. Then $f_!$ is well-defined and so $f_! : \partial_{\mathbf{C}}^{-1}(X) \to \partial_{\mathbf{C}}^{-1}(Y)$ is a functor clearly. We have:

Proposition 4.2.11 *Let* $\mathbf{C}$ *be a range restriction category and* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *the forgetful range restriction fibration. Then* $f : X \to Y$ *is total if and only if there is an adjunction:*

$$\partial_{\mathbf{C}}^{-1}(Y) \underset{f^*}{\overset{f_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \partial_{\mathbf{C}}^{-1}(X)$$

PROOF: "⇒" For any total map $f : X \to Y$ in $\mathbf{C}$ and $(X, e_X) \in \partial_{\mathbf{C}}^{-1}(X)$ and $(Y, e_Y) \in \partial_{\mathbf{C}}^{-1}(Y)$, we have

$$\begin{aligned}\widehat{fe_X} \le e_Y &\Leftrightarrow \overline{fe_X} \le \overline{e_Y fe_X} \text{ (by Lemma 4.2.10)} \\ &\Leftrightarrow e_X \le \overline{e_Y f} e_X \text{ (by [}\mathbf{R.3}\text{] and } \overline{f} = 1) \\ &\Leftrightarrow e_X = \overline{e_Y f} e_X e_X = \overline{e_Y f} e_X \\ &\Leftrightarrow e_X \le \overline{e_Y f}.\end{aligned}$$

Hence

$$\frac{\widehat{fe_X} \le e_Y}{\overline{\overline{e_X \le \overline{e_Y f}}}},$$

and therefore

$$\frac{f_!(X, e_X) \to (Y, e_Y)}{(X, e_X) \to f^*(Y, e_Y)} \; .$$

Thus, $f_! \dashv f^*$, as desired.

"$\Leftarrow$" If $f_! \dashv f^*$, then f^* preserves the top element of $\partial_{\mathbf{C}}^{-1}(Y)$ and this happens if and only if f is total. □

Now we recall that a range restriction fibration $\delta : \mathbf{D} \to \mathbf{C}$ gives rise to the inverse image functor $f^* : \delta^{-1}(Y) \to \delta^{-1}(X)$. By Proposition 4.2.8, $\delta \cong (\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C})$ for some range restriction category $\mathbf{C}$ and each fiber $\delta^{-1}(X) \cong \partial_{\mathbf{C}}^{-1}(X)$. Hence we also have the direct image functor $f_! : \delta^{-1}(X) \to \delta^{-1}(Y)$ sending $D_1 \leq D_2$ in $\delta^{-1}(X)$ to $w_{\widehat{f\varepsilon_{D_1}}} \leq w_{\widehat{f\varepsilon_{D_2}}}$ in $\delta^{-1}(Y)$, where $\omega_{\widehat{f\varepsilon_D}} = \omega_{\varepsilon_{\omega_{f\varepsilon_D}}}$. By Propositions 4.2.8 and 4.2.11, we have:

Proposition 4.2.12 *Let $\delta : \mathbf{D} \to \mathbf{C}$ be a range restriction fibration. Then $f : X \to Y$ is total if and only if there is an adjunction:*

$$\delta^{-1}(Y) \underset{f^*}{\overset{f_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \delta^{-1}(X)$$

where $f_!, f^*$ are direct and inverse image functors, respectively.

Chapter 5

The Free Range Restriction Categories over Directed Graphs

The objectives of this chapter are to construct the free range restriction categories over directed graphs and to give a source of range restriction categories whose word problems are decidable as well as some explicit examples of range stable meet semilattice fibrations. We first introduce the notions of *based directed graphs, based trees, based trees over a based directed graph,* and *deterministic trees.* Then we show that the deterministic based trees over a directed graph G give rise to a G^*-indexed category $dbTree : (G^*)^{\mathrm{op}} \to \mathbf{dbTree}(G)$. Finally, we apply the poset collapse to the indexed category $dbTree$ to obtain the desired free range stable meet semilattice fibration over G^*, which turns out to produce the free range restriction category over G.

5.1 Based Direct Graphs and Based Trees

In this section, we introduce the definitions of based directed graphs, based trees, and based trees over a based directed graph and their properties.

5.1.1 Definitions of Based Direct Graphs and Based Trees

Recall that a (*finite*) *directed graph* is an object of a functor category $(\mathbf{Set}_f)^{\mathbf{2}}$, where $\mathbf{2}$ is the category displayed by $\bullet \overset{d}{\underset{c}{\rightrightarrows}} \bullet$. Explicitly, a *directed graph* G consists of a finite set $V(G)$ of vertices, a finite set $E(G)$ of edges, and two maps

$$E(G) \underset{\partial_1^G}{\overset{\partial_0^G}{\rightrightarrows}} V(G)$$

Let $v_1, v_2 \in V(G)$. A *path* from vertex v_1 to v_2 is a list of finite distinct edges $[m_1, \cdots, m_k]$ such that $\partial_0^G(m_1) = v_1$, $\partial_1^G(m_i) = \partial_0^G(m_{i+1}), i = 1, \cdots, k-1$, and $\partial_1^G(m_k) = v_2$. A *undirected path* between vertices v_1 and v_2 in G is a *oriental path* $[m_1^{l_1}, \cdots, m_k^{l_k}]$ such that $\partial_0^G(m_1^{l_1}) = v_1, \partial_1^G(m_i^{l_i}) = \partial_0^G(m_{i+1}^{l_{i+1}}), i = 1, \cdots, k-1$, and $\partial_1^G(m_k^{l_k}) = v_2$ for some $l_1, \cdots, l_k \in \{-1, 1\}$, where $m_1, \cdots, m_k$ are distinct edges in G and $\partial_i(m_j^1) = \partial_i(m_j)$ and $\partial_i(m_j^{-1}) = \partial_{1-i}(m_j)$, $i \in \{0, 1\}$ and $j \in \{1, \cdots, k\}$.

A *tree* is a finite directed graph in which there is a unique undirected path between any two vertices. A *directed tree* is a finite directed graph in which there is a vertex $r \in V(G)$ such that for any $x \in V(G)$ there exists a unique path from r to x.

Let G and H be two directed graphs. Clearly, a *directed graph map* from G to H is a map in the functor category $(\mathbf{Set}_f)^\mathbf{2}$. Explicitly, a *directed graph map* $\nu : G \to H$ is pair $\nu = (\nu_E, \nu_V)$ of maps: $\nu_E : E(G) \to E(H)$ and $\nu_V : V(G) \to V(H)$ such that

$$\begin{array}{ccc} E(G) & \xrightarrow{\nu_E} & E(H) \\ \partial_0^G \downdownarrows \partial_1^G & & \partial_0^H \downdownarrows \partial_1^H \\ V(G) & \xrightarrow{\nu_V} & V(H) \end{array}$$

commutes serially. We shall denote the category $(\mathbf{Set}_f)^\mathbf{2}$ of directed graphs and directed graph maps by **Graph**.

Given a directed graph G, we can form the *path category* G^*: the objects of G^* are the same as $V(G)$ but the maps of G^* are lists of edges in G which juxtapose:

if $[m_1, \cdots, m_k]$ is a juxtaposing list from A to B in G then $(A, [m_1, \cdots, m_k], B) : A \to B$ is a map in G^*, the composition is given by concatenation and the identities are given by the empty paths: $1_A = (A, [\,], A)$. We can easily extend ∂_0 and ∂_1 from $E(G)$ to G^*.

Examples

1. Let G be given by $V(G) = \emptyset$ and $E(G) = \emptyset$. Then G is a directed graph (tree), which we shall denote by $\mathbf{0}$.

2. Let G be given by $V(G) = \{*\}$ and $E(G) = \emptyset$. Then G is a directed graph (tree), which we shall denote by $*$.

3. Let G have $V(G) = \{*\}$ and $E(G) = \{+\}$ and $\partial_0(+) = \partial_1(+) = *$. Then G is a directed graph (tree), which we shall denote by $\mathbf{1}$.

Definition 5.1.1 *A* based tree (based directed graph) T *is a tree (directed graph)* T *with a selected vertex* b_T*, namely, a map* $b_T : * \to T$*, denoted by* (T, b_T)*. A* based tree (based directed graph) map $\nu : (T, b_T) \to (T', b_{T'})$ *is a directed graph map* $\nu : T \to T'$ *such that*

$$\begin{array}{ccc} & * & \\ {\scriptstyle b_T} \swarrow & & \searrow {\scriptstyle b_{T'}} \\ T & \xrightarrow{\nu} & T' \end{array}$$

commutes.

Based trees and based tree maps form a category **bTree**. Also, based directed graph and based directed graph maps form a category **bGraph**. Obviously, **bTree** is a full subcategory of **bGraph**.

Examples

1. $(*,*)$ is the initial based directed graph (tree).

2. $(\mathbf{1},*)$ is the final based directed graph (tree).

5.1.2 Functors $(-)^+$ and $(-)^-$

If (T, b_T) is a based directed graph (based tree), then we can form a based directed graph (based tree) $(T, b_T)^+ = (T^+, *)$ by the following data:

$$V(T^+) = V(T) \sqcup \{*\}, E(T^+) = E(T) \sqcup \{+\},$$

and

$$\partial_0, \partial_1 : E(T^+) \to V(T^+)$$

given by

$$\partial_0(e) = \begin{cases} \partial_0(e), & \text{if } e \in E(T), \\ *, & \text{otherwise,} \end{cases}$$

$$\partial_1(e) = \begin{cases} \partial_1(e), & \text{if } e \in E(T), \\ b_T, & \text{otherwise.} \end{cases}$$

Moreover, if $\nu : (T, b_T) \to (T', b_{T'})$ is a based directed graph (based tree) map, then we have a based directed graph (based tree) map $\nu^+ : (T, b_T)^+ \to (T', b_{T'})^+$ which extends $\nu : (T, b_T) \to (T', b_{T'})$ by sending $+ : * \to b_T$ to $+ : * \to b_{T'}$. Clearly, we have:

Lemma 5.1.2 $(-)^+$: **bGraph** $\to$ **bGraph**, *sending* $\nu : (T, b_T) \to (T', b_{T'})$ *to*

$\nu^+ : (T, b_T)^+ \to (T', b_{T'})^+$, *is a functor.*

Lemma 5.1.3 *Let* $h : A \to B$ *be an edge in a directed graph* G. *Then there is a based directed graph map* $\check{h} : (G, B)^+ \to (G, A)$.

PROOF: Define $\check{h} : (G, B)^+ \to (G, A)$ by $\check{h}(e) = e$ if $e \in E(G)$ and $\check{h}(+) = h$. Clearly, $\check{h} : (G, B)^+ \to (G, A)$ is a based directed graph map. □

Similarly, given a based directed graph (based tree) (T, b_T), we can form a based directed graph (based tree) $(T, b_T)^- = (T^-, *)$ by the following data:

$$V(T^-) = V(T) \sqcup \{*\}, E(T^-) = E(T) \sqcup \{-\},$$

and

$$\partial_0, \partial_1 : E(T^-) \to V(T^-)$$

given by

$$\partial_0(e) = \begin{cases} \partial_0(e), & \text{if } e \in E(T), \\ b_T, & \text{otherwise,} \end{cases}$$

$$\partial_1(e) = \begin{cases} \partial_1(e), & \text{if } e \in E(T), \\ *, & \text{otherwise.} \end{cases}$$

If $\nu : (T, b_T) \to (T', b_{T'})$ is a based directed graph (based tree) map, then we have a based directed graph (based tree) map $\nu^- : (T, b_T)^- \to (T', b_{T'})^-$ which extends $\nu : (T, b_T) \to (T', b_{T'})$ by sending $- : b_T \to *$ to $- : b_{T'} \to *$. Clearly, we have:

Lemma 5.1.4 $(-)^- : \mathbf{bGraph} \to \mathbf{bGraph}$ *is a functor.*

Lemma 5.1.5 *Let $h : A \to B$ be an edge in a directed graph G. Then there is a based directed graph map $\hat{h} : (G, A)^- \to (G, B)$.*

PROOF: Define $\hat{h} : (G, A)^- \to (G, B)$ by $\hat{h}(e) = e$ if $e \in E(G)$ and $\hat{h}(-) = h$. Clearly, $\hat{h} : (G, A)^- \to (G, B)$ is a based directed graph map. $\square$

Examples

1. $*^+ = * \xrightarrow{+} *'$, $*^- = *' \xrightarrow{-} *$.

2. $(*^+)^+ = * \xrightarrow{+} *' \xrightarrow{+'} *''$, $(*^-)^- = *'' \xrightarrow{-'} *' \xrightarrow{-} *$.

5.1.3 Trees over Directed Graphs

A *tree T over a directed graph G* consists of a tree T and a directed graph map $\nu : T \to G$. The trees over G form a category $\mathbf{Tree}(G)$ with trees over G as objects and with a direct graph map $\tau : T_1 \to T_2$ such that

$$\begin{array}{ccc} T_1 & \xrightarrow{\tau} & T_2 \\ & {\scriptstyle \nu_1} \searrow \quad \swarrow {\scriptstyle \nu_2} & \\ & G & \end{array}$$

commutes as a map from $\nu_1 : T_1 \to G$ to $\nu_2 : T_2 \to G$. Its composition and identities are obvious.

Let (G, A) be a based directed graph. A *based tree $((T, b_T), \nu_T)$ over (G, A)* consists of a based tree (T, b_T) and a based directed graph map $\nu_T : (T, b_T) \to (G, A)$. A *based tree map $\tau : ((T, b_T), \nu_T) \to ((T', b_{T'}), \nu_{T'})$* is a based directed graph map

$\tau : (T, b_T) \to (T', b_{T'})$ such that

$$\begin{array}{ccc} (T, b_T) & \xrightarrow{\tau} & (T', b_{T'}) \\ & {}_{\nu_T}\searrow \quad \swarrow_{\nu_{T'}} & \\ & (G, A) & \end{array}$$

commutes in **bTree**.

Let $\alpha : G \to H$ be a directed graph map and $((T, b_T), \nu_T) \in \mathbf{bTree}(G, A)$. Then $((T, b_T), \alpha\nu_T) \in \mathbf{bTree}(H, \alpha(A))$.

All based trees over the based directed graph (G, A) and maps between them form a category $\mathbf{bTree}(G, A)$. This category always has finite coproducts:

Lemma 5.1.6 *Let (G, A) be a based directed graph. Then* $\mathbf{bTree}(G, A)$ *has finite coproducts.*

PROOF: Clearly, $((A, A), 1_A)$ is the initial object in $\mathbf{bTree}(G, A)$. So, it suffices to show that $\mathbf{bTree}(G, A)$ has binary coproducts. For any $((T, b_T), \nu_T), ((S, b_S), \nu_S) \in \mathbf{bTree}(G, A)$, we form a based tree $(T + S, b_{T+S})$ by just identifying b_T and b_S, called by b_{T+S}, namely,

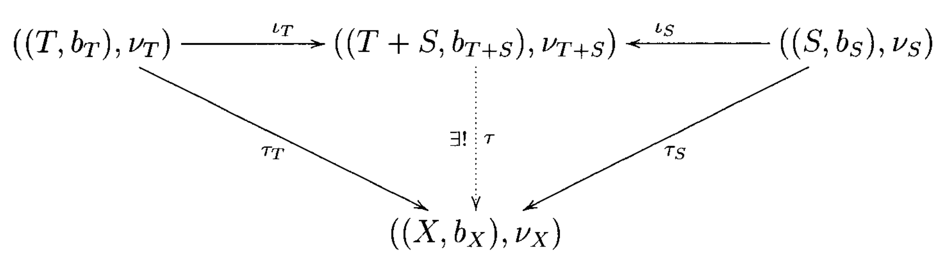
$$V(T + S) = (V(T) \sqcup V(S) \setminus \{b_T, b_S\}) \sqcup \{b_{T+S}\},$$

and

$$E(T + S) = \{e \in E(T) \sqcup E(S) \mid \text{if } \partial_i(e) \in \{b_T, b_S\} \text{ rename } \partial_i(e) \text{ to } b_{T+S}, i = 0, 1\}.$$

Clearly, $(T + S, b_{T+S})$ is a based directed graph. For any $v_1, v_2 \in V(T + S)$, if both

v_1 and v_2 are in $V(T)$ or $V(S)$, then there is a unique undirected path between v_1 and v_2 since both T and S are based trees. If $v_i \in V(T)$ and $v_j \in V(S)$, $i, j \in \{1, 2\}, i \neq j$, then there is a unique undirected path $[m_1, \cdots, m_k]$ connecting v_i and b_{T+S} and a unique undirected path $[n_1, \cdots, n_l]$ connecting v_j and b_{T+S} and so a unique undirected path $[m_1, \cdots, m_k, n_1, \cdots, n_l]$ connecting v_i and v_j. Therefore, $(T + S, b_{T+S})$ is a based tree. Define $\nu_{T+S} : (T + S, b_{T+S}) \to (G, A)$ by sending $v \in V(T)$ to $\nu_T(v)$, $v \in V(S)$ to $\nu_S(v)$, b_{T+S} to $\nu_T(b_T)$, and $e \in E(T)$ to $\nu_T(e)$, $e \in E(S)$ to $\nu_S(e)$. Then $(((T + S, b_{T+S}), \nu_{T+S}), \iota_1, \iota_2)$ is a coproduct of $((T, b_T), \nu_T)$ and $((S, b_S), \nu_S)$ in **bTree**(G, A), where $\iota_T : ((T, b_T), \nu_T) \to ((T + S, b_{T+S}), \nu_{T+S})$ and $\iota_S : ((S, b_S), \nu_S) \to ((T + S, b_{T+S}), \nu_{T+S})$ are the embeddings. In fact, for any maps $\tau_T : ((T, b_T), \nu_T) \to ((X, b_X), \nu_X)$ and $\tau_S : ((S, b_S), \nu_S) \to ((X, b_X), \nu_X)$ in **bTree**(G, A), there is a unique map $\tau : ((T + S, b_{T+S}), \nu_{T+S}) \to ((X, b_X), \nu_X)$ given by sending $v \in V(T)$ to $\tau_T(v)$, $v \in V(S)$ to $\tau_S(v)$, b_{T+S} to $\tau_T(b_T) = \tau_S(b_S) = b_X$, and sending $e \in E(T)$ to $\tau_T(e)$, $e \in E(S)$ to $\tau_S(e)$, such that



commutes in **bTree**(G, A). □

5.1.4 The Indexed Category $bTree : (G^*)^{\text{op}} \to$ **bTree**(G)

Given a directed graph G, we form the category of all **bTree**$(G, A), A \in V(G)$ and functors between them, denoted by **bTree**(G). In order to form the desired indexed category, we need the *change base functor*:

Lemma 5.1.7 *Let $h : A \to B$ be an edge in a directed graph G. Then there is a functor $\Box h : \mathbf{bTree}(G, B) \to \mathbf{bTree}(G, A)$.*

PROOF: For any map $\tau : ((T, b_T), \nu_T) \to ((T', b_{T'}), \nu_{T'})$ in $\mathbf{bTree}(G, B)$, we have a commutative diagram

$$\begin{array}{ccc} (T, b_T) & \xrightarrow{\quad \tau \quad} & (T', b_{T'}) \\ & {\scriptstyle \nu_T} \searrow \quad \swarrow {\scriptstyle \nu_{T'}} & \\ & (G, B) & \end{array}$$

in $\mathbf{bTree}$. Applying the functor $(-)^+$ to the last diagram, we get a commutative diagram

$$\begin{array}{ccc} (T, b_T)^+ & \xrightarrow{\quad \tau^+ \quad} & (T', b_{T'})^+ \\ & {\scriptstyle \nu_T^+} \searrow \quad \swarrow {\scriptstyle \nu_{T'}^+} & \\ & (G, B)^+ & \end{array}$$

in $\mathbf{bTree}$ and so a commutative diagram

$$\begin{array}{ccc} (T, b_T)^+ & \xrightarrow{\quad \tau^+ \quad} & (T', b_{T'})^+ \\ & {\scriptstyle \check{h}\nu_T^+} \searrow \quad \swarrow {\scriptstyle \check{h}\nu_{T'}^+} & \\ & (G, A) & \end{array}$$

in $\mathbf{bTree}$. Hence there is a map

$$\tau^+ : ((T, b_T)^+, \check{h}\nu_T^+) \to ((T', b_{T'})^+, \check{h}\nu_{T'}^+)$$

in $\mathbf{bTree}(G,A)$ and therefore a functor $\Box h : \mathbf{bTree}(G,B) \to \mathbf{bTree}(G,A)$ taking $\tau : ((T,b_T),\nu_T) \to ((T',b_{T'}),\nu_{T'})$ to $\tau^+ : ((T,b_T)^+, \check{h}\nu_T^+) \to ((T',b_{T'})^+, \check{h}\nu_{T'}^+)$. $\Box$

By Lemma 5.1.7, we have the *change base* functor:

Lemma 5.1.8 *Let $h : A \to B$ be a path in a directed graph G. Then there is a functor $\Box h : \mathbf{bTree}(G,B) \to \mathbf{bTree}(G,A)$.*

Proof: Assume that $h = [m_1, \cdots, m_k]$ is a path from A to B. By Lemma 5.1.7 we have a functor $\Box m_i : \mathbf{bTree}(G, \partial_1(m_i)) \to \mathbf{bTree}(G, \partial_0(m_i))$, for each $i \in \{1, \cdots, k\}$. Hence we have a functor $\Box h = (\Box m_k) \cdots (\Box m_1) : \mathbf{bTree}(G,B) \to \mathbf{bTree}(G,A)$. $\Box$

Similarly, for any map $\tau : ((T,b_T),\nu_T) \to ((T',b_{T'}),\nu_{T'})$ in $\mathbf{bTree}(G,A)$, we have a commutative diagram

$$\begin{array}{ccc} (T,b_T) & \xrightarrow{\quad\tau\quad} & (T',b_{T'}) \\ & {\scriptstyle \nu_T}\searrow \quad \swarrow{\scriptstyle \nu_{T'}} & \\ & (G,A) & \end{array}$$

in $\mathbf{bTree}$. Applying the functor $(-)^-$, we have a commutative diagram

$$\begin{array}{ccc} (T,b_T)^- & \xrightarrow{\quad\tau^-\quad} & (T',b_{T'})^- \\ & {\scriptstyle \nu_T^-}\searrow \quad \swarrow{\scriptstyle \nu_{T'}^-} & \\ & (G,A)^- & \end{array}$$

in **bTree** and so a commutative diagram

$$\begin{array}{ccc} (T, b_T)^- & \xrightarrow{\tau^-} & (T', b_{T'})^- \\ & {\scriptstyle \hat{h}\nu_T^-} \searrow \quad \swarrow {\scriptstyle \hat{h}\nu_{T'}^-} & \\ & (G, B) & \end{array}$$

in **bTree**. Hence there is a map

$$\tau^- : ((T, b_T)^-, \hat{h}\nu_T^-) \to ((T', b_{T'})^-, \hat{h}\nu_{T'}^-)$$

in $\mathbf{bTree}(G, A)$ and therefore a functor $\Diamond h : \mathbf{bTree}(G, A) \to \mathbf{bTree}(G, B)$ taking $\tau : ((T, b_T), \nu_T) \to ((T', b_{T'}), \nu_{T'})$ to $\tau^- : ((T, b_T)^-, \hat{h}\nu_T^-) \to ((T', b_{T'})^-, \hat{h}\nu_{T'}^-)$. As in Lemma 5.1.8, we have:

Lemma 5.1.9 *Let $h : A \to B$ be a path in a directed graph G. Then $\Diamond h : \mathbf{bTree}(G, A) \to \mathbf{bTree}(G, B)$ is a functor.*

By Lemma 5.1.8, we immediately have:

Proposition 5.1.10 *bTree : $(G^*)^{\mathrm{op}} \to \mathbf{bTree}(G)$, sending each $h : B \to A$ in $(G^*)^{\mathrm{op}}$ to a functor $\Box h : \mathbf{bTree}(G, B) \to \mathbf{bTree}(G, A)$ in $\mathbf{bTree}(G)$, is an indexed category, in which there is a functor $\Diamond h : \mathbf{bTree}(G, A) \to \mathbf{bTree}(G, B)$ for each path $h : A \to B$ in G.*

5.2 Range Stable Meet Semilattice Fibrations over Directed Graphs

The goal of this section is to construct the range stable meet semilattice fibration over a given directed graph using deterministic based trees.

5.2.1 Deterministic Based Trees

In order to form a range stable meet semilattice fibration, we introduce the notion of *deterministic based trees.* Throughout this subsection, (G, A) denotes a based directed graph.

For each edge $e \in E(G)$, we only assume that $[e^{-1}, e]$ is a path from $\partial_1(e)$ to $\partial_1(e)$: $e^{-1}e = (\partial_1(e), [\,], \partial_1(e))$. For any given $((T, b_T), \nu_T) \in \mathbf{bTree}(G, A)$, we define a relation $\sim$ on $V(T)$ by

$$v_1 \sim v_2 \quad \Leftrightarrow \quad \text{there is a undirected path } [p_1, \cdots, p_k] \text{ from } v_1 \text{ to } v_2 \text{ such that}$$
$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2)) \text{ in } G^*.$$

Lemma 5.2.1 *$\sim$ is an equivalence relation on $V(T)$.*

PROOF: For any $v \in V(T)$, there is a undirected path $(v, [\,], v) : v \to v$ such that $\nu_T(v, [\,], v) = (v, [\,], v)$. Hence $v \sim v$ and therefore $\sim$ is reflexive.

If $v_1 \sim v_2$, then there is a undirected path $[p_1, \cdots, p_k]$ from v_1 to v_2 such that

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2)),$$

and so there is a undirected path $[p_k, \cdots, p_1]$ from v_2 to v_1 such that

$$\nu_T(p_k) \cdots \nu_T(p_1) = (\nu_T(v_2), [\,], \nu_T(v_1)).$$

Hence $v_2 \sim v_1$ and therefore $\sim$ is symmetric.

If $v_1 \sim v_2$ and $v_2 \sim v_3$, then there are undirected paths $[p_1, \cdots, p_k]$ from v_1 to v_2 and $[q_1, \cdots, q_l]$ from v_2 to v_3 such that

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2))$$

and

$$\nu_T(q_1) \cdots \nu_T(q_l) = (\nu_T(v_2)), [\,], \nu_T(v_3)).$$

If $\{p_1, \cdots, p_k\} \cap \{q_1^{-1}, \cdots, q_l^{-1}\} = \emptyset$, then there is a undirected path

$$[p_1, \cdots, p_k, q_1, \cdots, q_l]$$

from v_1 to v_3 such that

$$\nu_T(p_1) \cdots \nu_T(p_k)\nu_T(q_1) \cdots \nu_T(q_l) = (\nu_T(v_1), [\,], \nu_T(v_3)).$$

If $\{p_1, \cdots, p_k\} \cap \{q_1^{-1}, \cdots, q_l^{-1}\} = \{c_1, \cdots, c_m\} \neq \emptyset$, then

$$[p_1, \cdots, p_k] = [p_1, \cdots, p_{k-m}, c_1, \cdots, c_m]$$

and

$$[q_1, \cdots, q_l] = [c_m^{-1}, \cdots, c_1^{-1}, q_{l-m+1}, \cdots, q_l].$$

It is easy to check that

$$[p_1, \cdots, p_{k-m}, q_{l-m+1}, \cdots, q_l]$$

is a undirected path from v_1 to v_3, which is obtained by deleting the common part of $[p_1, \cdots, p_k]$ and $[q_1, \cdots, q_l]$, such that

$$\nu_T(p_1) \cdots \nu_T(p_{k-m}) \nu_T(q_{l-m+1}) \cdots \nu_T(q_l) = (\nu_T(v_1), [\,], \nu_T(v_3)).$$

Hence $v_1 \sim v_3$ always and therefore $\sim$ is transitive. □

This equivalence relation $\sim$ introduces a relation $\sim_E$ on $E(T)$:

$$e_1 \sim_E e_2 \Leftrightarrow \partial_0(e_1) \sim \partial_0(e_2) \text{ and } \partial_1(e_1) \sim \partial_1(e_2).$$

Clearly, $\sim_E$ is an equivalence relation on $E(T)$ since $\sim$ is an equivalence relation.

Definition 5.2.2 $((T, b_T), \nu_T) \in \mathbf{bTree}(G, A)$ *is called* deterministic *if* $e_1 = e_2$ *whenever* $e_1, e_2 \in E(T)$ *are such that* $e_1 \sim_E e_2$.

All deterministic based trees over (G, A) and based directed graph maps between them form a category $\mathbf{dbTree}(G, A)$. We shall see that $\mathbf{dbTree}(G, A)$ turns out to be a reflective subcategory of $\mathbf{bTree}(G, A)$.

Now we define a tree $T/\sim$ by

$$V(T/\sim) = V(T)/\sim, E(T/\sim) = E(T)/\sim_E$$

and

$$\partial_0([e]) = [\partial_0(e)], \partial_1([e]) = [\partial_1(e)].$$

If $e_1 \sim e_2$, then $\partial_i(e_1) \sim \partial_i(e_2)$ and so $[\partial_i(e_1)] = [\partial_i(e_2)], i = 0, 1$. Hence ∂_0 and ∂_1 of $T/\sim$ are well-defined. Therefore, we have a based tree $(T/\sim, [b_T])$. Define

$$\nu_{T/\sim} : (T/\sim, [b_T]) \to (G, A)$$

by sending $[v] \in V(T/\sim)$ to $\nu_T(v)$ and $[e] \in E(T/\sim)$ to $\nu_T(e)$. If $[v_1] = [v_2]$, then $v_1 \sim v_2$ and so there is a undirected path $[p_1, \cdots, p_k]$ from v_1 to v_2 such that

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2))$$

and so $\nu_T(v_1) = \nu_T(v_2)$. If $[e_1] = [e_2]$, then $\partial_0(e_1) \sim \partial_0(e_2)$ and $\partial_1(e_1) \sim \partial_1(e_2)$ and so there are undirected paths $[p_1, \cdots, p_k]$ from $\partial_0(e_1)$ to $\partial_0(e_2)$ and $[q_1, \cdots, q_l]$ from $\partial_1(e_1)$ to $\partial_1(e_2)$ such that

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(\partial_0(e_1)), [\,], \nu_T(\partial_0(e_2)))$$

and

$$\nu_T(q_1) \cdots \nu_T(q_l) = (\nu_T(\partial_1(e_1)), [\,], \nu_T(\partial_1(e_2))).$$

Since there is a unique undirected path from $\partial_1(e_1)$ to $\partial_1(e_2)$, we have $\nu_T(e_1) = \nu_T(e_2)$. Hence $\nu_{T/\sim}$ is a well-defined directed graph map and therefore we have a based tree $((T/\sim, [b_T]), \nu_{T/\sim})$ over (G, A), which we shall denote by $Det((T, b_T), \nu_T)$.

An easy observation is:

Lemma 5.2.3 *Let* $((T, b_T), \nu_T) \in \mathbf{bTree}(G, A)$, $v_1, v_2 \in V(T)$ *and* $e_1, e_2 \in E(T)$. *Then*

(*i*) $[v_1] \sim [v_2]$ *in* $Det((T, b_T), \nu_T) \Leftrightarrow v_1 \sim v_2$ *in* $((T, b_T), \nu_T) \Leftrightarrow [v_1] = [v_2]$ *in* $Det((T, b_T), \nu_T)$.

(*ii*) $[e_1] \sim [e_2]$ *in* $Det((T, b_T), \nu_T) \Leftrightarrow e_1 \sim e_2$ *in* $((T, b_T), \nu_T) \Leftrightarrow [e_1] = [e_2]$ *in* $Det((T, b_T), \nu_T)$.

PROOF: (*i*) By the definition of $Det((T, b_T), \nu_T)$, clearly, $v_1 \sim v_2 \Rightarrow [v_1] = [v_2] \Rightarrow [v_1] \sim [v_2]$ and $v_1 \sim v_2$ in $((T, b_T), \nu_T) \Leftrightarrow [v_1] = [v_2]$ in $Det((T, b_T), \nu_T)$. So it suffices to prove that $[v_1] \sim [v_2] \Rightarrow v_1 \sim v_2$.

If $[v_1] \sim [v_2]$, then there is a undirected path $[[p_1], \cdots, [p_k]]$ in $Det((T, b_T), \nu_T)$ such that

$$\nu_{T/\sim}([p_1]) \cdots \nu_{T/\sim}([p_k]) = (\nu_{T/\sim}([v_1]), [\,], \nu_{T/\sim}([v_2])),$$

and so

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2)).$$

Since $[[p_1], \cdots, [p_k]]$ is a undirected path in $Det((T, b_T), \nu_T)$, for each $1 \leq i \leq k_1$, $\partial_1([p_i]) = \partial_0([p_{i+1}])$. Hence $[\partial_1(p_i)] = [\partial_0(p_{i+1})]$ and therefore $\partial_1(p_i) \sim \partial_0(p_{i+1})$, for $i = 1, \cdots, k-1$. It follows that there exist undirected paths $[q_{i,1}, \cdots, q_{i,i_k}]$ from $\partial_1(p_i)$ to $\partial_0(p_{i+1})$ in $((T, b_T), \nu_T)$ such that

$$\nu_T(q_{i,1}) \cdots \nu_T(q_{i,i_k}) = (\nu_T(\partial_1(p_i)), [\,], \nu_T(\partial_0(p_{i+1}))), i = 1, \cdots, k-1.$$

Hence there is a undirected path $[s_1, \cdots, s_m]$ from v_1 to v_2, obtained from

$$\{p_1, q_{1,1}, \cdots, q_{1,i_1}, \cdots, p_{k-1}, q_{k-1,1}, \cdots, q_{k-1,i_{k-1}}, p_k\}$$

by deleting the common parts as we did in the proof of Lemma 5.2.1, such that

$$\nu_T(s_1)\cdots\nu_T(s_m) = (\nu_T(v_1), [\,], \nu_T(v_2))$$

and therefore $v_1 \sim v_2$, as desired.

(ii) Similarly, it suffices to show that $[e_1] \sim [e_2] \Rightarrow e_1 \sim e_2$. If $[e_1] \sim [e_2]$, then $\partial_i[e_1] \sim \partial_i[e_2]$ and so $[\partial_i(e_1)] \sim [\partial_i(e_2)], i = 0, 1$. By *(i)*, $\partial_i(e_1) \sim \partial_i(e_2), i = 0, 1$. Hence $e_1 \sim e_2$. □

Lemma 5.2.4 *$Det((T, b_T), \nu_T)$ is a deterministic tree.*

PROOF: Let $[e_1], [e_2] \in E(T/\sim)$ with $[e_1] \sim [e_2]$. Then, by Lemma 5.2.3, $e_1 \sim e_2$ and so $[e_1] = [e_2]$. Hence $Det((T, b_T), \nu_T)$ is a deterministic tree. □

If $\tau : ((T, b_T), \nu_T) \to ((S, b_S), \nu_S)$ is a directed graph map in $\mathbf{bTree}(G, A)$, then we define

$$Det(\tau) : Det((T, b_T), \nu_T) \to Det((S, b_S), \nu_S)$$

by sending $[v]$ to $[\tau(v)]$ and sending $[e]$ to $[\tau(e)]$. If $v_1 \sim v_2$, then there is a undirected path $[p_1, \cdots, p_k]$ from v_1 to v_2 such that

$$\nu_T(p_1)\cdots\nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2)).$$

Hence there is a undirected path $[q_1, \cdots, q_l]$ obtained from $[\tau(p_1), \cdots, \tau(p_k)]$ from $\tau(v_1)$ to $\tau(v_2)$ such that

$$\nu_X(q_1)\cdots\nu_X(g_l) = \nu_X(\tau(p_1))\cdots\nu_X(\tau(p_k)) = (\nu_X(\tau(v_1)), [\,], \nu_X(\tau(v_2)))$$

and so $\tau(v_1) \sim \tau(v_2)$. Hence $[\tau(v_1)] = [\tau(v_2)]$. Similarly, if $e_1 \sim_E e_2$, then $\tau(e_1) \sim_E \tau(e_2)$ and so $[\tau(e_1)] = [\tau(e_2)]$. Therefore, $Det(\tau)$ is a well-defined directed graph map. Clearly, we have:

Lemma 5.2.5 *Det : $\mathbf{bTree}(G, A) \to \mathbf{dbTree}(G, A)$, sending $\tau : ((T, b_T), \nu_T) \to ((S, b_S), \nu_S)$ to $Det(\tau) : Det((T, b_T), \nu_T) \to Det((S, b_S), \nu_S)$, is a functor.*

In the other direction, since each deterministic based tree is, of course, a based tree, we have an inclusion functor $\iota : \mathbf{dbTree}(G, A) \to \mathbf{bTree}(G, A)$. Now we are ready to prove:

Lemma 5.2.6 *Let (G, A) be a based directed graph. Then $\mathbf{dbTree}(G, A)$ is a fully reflective subcategory of $\mathbf{bTree}(G, A)$ and has finite coproducts.*

Proof: We shall prove that Det is a left adjoint functor of the inclusion ι : $\mathbf{dbTree}(G, A) \to \mathbf{bTree}(G, A)$.

For any $((T, b_T), \nu_T) \in \mathbf{bTree}(G, A)$, we define a based tree map $\eta_{((T,b_T),\nu_T)}$: $((T, b_T), \nu_T) \to Det((T, b_T), \nu_T)$ by taking v to $[v]$ and e to $[e]$, which serves as the unit of $Det_A \dashv \iota$. For any map $\tau : ((T, b_T), \nu_T) \to \iota((X, b_X), \nu_X)$ in $\mathbf{bTree}(G, A)$, we define $\tau^{\#} : Det((T, b_T), \nu_T) \to ((X, b_X), \nu_X)$ by sending $[v]$ to $\tau(v)$ and $[e]$ to $\tau(e)$.

If $v_1 \sim v_2$, then there is a undirected path $[p_1, \cdots, p_k]$ from v_1 to v_2 such that

$$\nu_T(p_1) \cdots \nu_T(p_k) = (\nu_T(v_1), [\,], \nu_T(v_2)),$$

and so there is a directed path $[q_1, \cdots, q_l]$ obtained from $[\tau(p_1), \cdots, \tau(p_k)]$ from $\tau(v_1)$ to $\tau(v_2)$ such that

$$\nu_X(q_1) \cdots \nu_X(q_l) = \nu_X\tau(p_1) \cdots \nu_X\tau(p_k) = (\nu_X\tau(v_1), [\,], \nu_X\tau(v_2)).$$

Since $((X, b_X), \nu_X)$ is deterministic, by Lemma 5.2.3 $\tau(v_1) = \tau(v_2)$.

Similarly, if $e_1 \sim_E e_2$, then $\tau(e_1) = \tau(e_2)$. Hence $\tau^\#$ is well-defined and therefore there is a unique map $\tau^\# : Det((T, b_T), \nu_T) \to ((X, b_X), \nu_X)$ in $\mathbf{dbTree}(G, A)$ such that

$$\begin{array}{ccc} ((T, b_T), \nu_T) & \xrightarrow{\eta_{((T,b_T),\nu_T)}} & \iota(Det((T, b_T), \nu_T)) \\ & \searrow^{\tau} & \downarrow{\scriptstyle \iota(\tau^\#)} \\ & & \iota((X, b_X), \nu_X) \end{array} \qquad \begin{array}{c} Det((T, b_T), \nu_T) \\ \vdots\, {\scriptstyle \tau^\#} \\ ((X, b_X), \nu_X) \end{array}$$

commutes. Thus, $\mathbf{dbTree}_A(G)$ is a fully reflective subcategory of $\mathbf{bTree}(G, A)$. $\square$

By Lemma 5.2.6, *Det* is a left adjoint of the inclusion functor. Hence *Det* preserves finite coproducts. So,

Lemma 5.2.7 *The functor Det* : $\mathbf{bTree}(G, A) \to \mathbf{dbTree}(G, A)$ *preserves finite coproducts.*

By Lemma 5.1.6, $\mathbf{bTree}(G, A)$ has finite coproducts. The binary coproducts of $\mathbf{dbTree}(G, A)$ can be created by using the functor *Det*: take the binary coproducts in $\mathbf{bTree}(G, A)$ first, then apply the functor *Det*.

Let $\mathbf{dbTree}(G)$ be the category of all $\mathbf{dbTree}(G,A), A \in V(G)$ and functors between them. By Proposition 5.1.10, $bTree : (G^*)^{\mathrm{op}} \to \mathbf{bTree}(G)$, sending each $h : B \to A$ in $(G^*)^{\mathrm{op}}$ to a functor $\Box h : \mathbf{bTree}(G,B) \to \mathbf{bTree}(G,A)$, is an indexed category, in which there is a functor $\Diamond h : \mathbf{bTree}(G,A) \to \mathbf{bTree}(G,B)$ for each path $h : A \to B$ in G. If $((T,b_T),\nu_T) \in \mathbf{bTree}(G,A)$ is *deterministic*, then so is $\Box h((T,b_T),\nu_T) = ((T,b_T)^+, \check{h}\nu_T)$ in $\mathbf{bTree}(G,B)$. In fact, let $h_1, h_2 \in E((T,b_T)^+)$ be such that $h_1 \sim h_2$. Since $((T,b_T),\nu_T) \in \mathbf{bTree}(G,A)$ is deterministic, $h_1 = h_2 = +$ or $h_1, h_2 \in E(T)$. On the other hand, $h_1, h_2 \in E(T)$ and $h_1 \sim_E h_2$ imply $h_1 = h_2$ since $((T,b_T),\nu_T)$ is deterministic, and so $h_1 = h_2$ in all cases. Hence $dbTree : (G^*)^{\mathrm{op}} \to \mathbf{dbTree}(G)$, taking $h : B \to A$ in $(G^*)^{\mathrm{op}}$ to

$$\Box h : \mathbf{dbTree}(G,B) \to \mathbf{dbTree}(G,A),$$

is a functor.

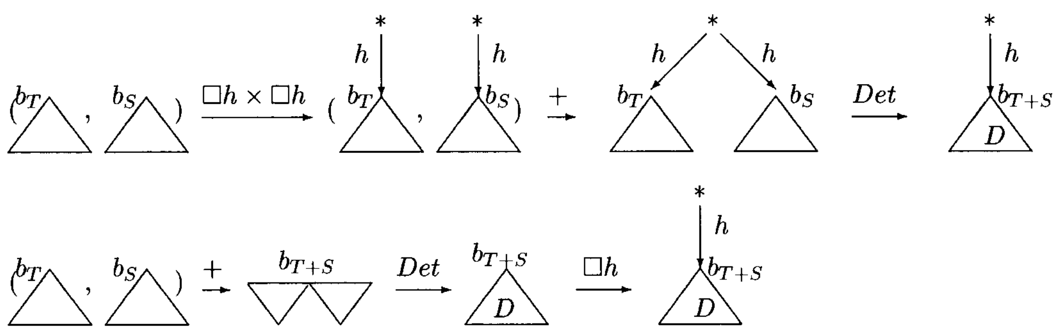
It is easy to check that for any path $h : A \to B$,

$$\begin{array}{ccc}
\mathbf{dbTree}(G,B) \times \mathbf{dbTree}(G,B) & \xrightarrow{\Box h \times \Box h} & \mathbf{dbTree}(G,A) \times \mathbf{dbTree}(G,A) \\
{\scriptstyle +}\big\downarrow & & \big\downarrow{\scriptstyle +} \\
\mathbf{bTree}(G,B) & & \mathbf{bTree}(G,A) \\
{\scriptstyle Det}\big\downarrow & & \big\downarrow{\scriptstyle Det} \\
\mathbf{dbTree} & \xrightarrow{\qquad\qquad \Box h \qquad\qquad} & \mathbf{dbTree}(G,A)
\end{array}$$

commutes, since for any $((T, b_T), \nu_T), ((S, b_S), \nu_S) \in \mathbf{dbTree}(G, B)$,

$$
\begin{aligned}
\Box h(Det((T, b_T), \nu_T) + ((S, b_S), \nu_S)) &= \Box h((Det(T+S), [b_{T+S}]), \nu_{T+S}) \\
&= (((Det(T+S))^+, *), \check{h}(\nu_{T+S})) \\
&= ((Det(T^+ + S^+), *), \check{h}\nu_T^+ + \check{h}\nu_S^+) \\
&= ((T^+, *), \check{h}\nu_T^+) + ((S^+, *), \check{h}\nu_S^+) \\
&= \Box h((T, b_T), \nu_T) + \Box h((S, b_S), \nu_S)
\end{aligned}
$$

as displayed graphically:



Then $\Box h : \mathbf{dbTree}(G, B) \to \mathbf{dbTree}(G, A)$ preserves binary coproducts.

On the other hand, $\triangle h = (Det)(\Diamond h)(\iota) : \mathbf{dbTree}(G, A) \to \mathbf{dbTree}(G, B)$ is a functor, for each path $h : A \to B$ in G.

Lemma 5.2.8 *dbTree* : $(G^*)^{\mathrm{op}} \to \mathbf{dbTree}(G)$, *sending each* $h : A \to B$ *to a binary coproducts preserving functor* $\Box h : \mathbf{dbTree}(G, A) \to \mathbf{dbTree}(G, A)$, *is an indexed category, in which there is a functor*

$$\triangle h = (Det_A)(\Diamond h)(\iota) : \mathbf{dbTree}(G, A) \to \mathbf{dbTree}(G, B)$$

for each path $h : A \to B$ in G.

5.2.2 Poset Collapse

Recall that a small category $\mathbf{C}$ with finite coproducts is a preorder with the binary relation $\leq$ given by

$$X \leq Y \Leftrightarrow \text{ there is a map } X \to Y.$$

Let $(\mathbf{Poset}(\mathbf{C}), \leq)$ be the *poset reflection* of $(\mathbf{C}, \leq)$. Then $(\mathbf{Poset}(\mathbf{C}), \leq)$ is a join semilattice with the join given by coproducts and with the bottom element given by the initial object of $\mathbf{C}$ and so, dually, $(\mathbf{Poset}(\mathbf{C}), \leq)^{\text{op}}$ is a meet semilattice.

Let $\mathbf{Cat}_+$ be the category of small categories with finite coproducts and functors preserving binary coproducts. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_+$ and let $F : \mathbf{C} \to \mathbf{D}$ be a binary coproducts preserving functor. Then $\mathbf{Poset}(F) : \mathbf{Poset}(\mathbf{C}) \to \mathbf{Poset}(\mathbf{D})$, sending $X \leq Y$ to $F(X) \leq F(Y)$, is a stable join semilattice homomorphism, and so $(\mathbf{Poset}(F))^{\text{op}} : (\mathbf{Poset}(\mathbf{C}))^{\text{op}} \to (\mathbf{Poset}(\mathbf{D}))^{\text{op}}$ is a stable meet semilattice homomorphism. Hence we have:

Lemma 5.2.9 *$Poset^{\text{op}} : \mathbf{Cat}_+ \to \mathbf{msLat}$, given by taking $F : \mathbf{C} \to \mathbf{D}$ to*

$$(\mathbf{Poset}(F))^{\text{op}} : (\mathbf{Poset}(\mathbf{C}))^{\text{op}} \to (\mathbf{Poset}(\mathbf{D}))^{\text{op}},$$

is a functor.

5.2.3 The Range Stable Meet Semilattice Fibration ∂_G

Let G be a directed graph and $h : A \to B$ a path in G. Then we have a binary coproducts preserving functor $\Box h : \mathbf{dbTree}(G, B) \to \mathbf{dbTree}(G, A)$ and a functor

$\triangle h : \mathbf{dbTree}(G, A) \to \mathbf{dbTree}(G, B)$. By Lemma 5.2.9, we have:

Lemma 5.2.10 *PosetdbTree* $: (G^*)^{\mathrm{op}} \to \mathbf{msLat}$, *sending* $h : B \to A$ *in* $(G^*)^{\mathrm{op}}$ *to*

$$h^* = (\mathbf{Poset}(\Box h))^{\mathrm{op}} : (\mathbf{Poset}(\mathbf{dbTree}(G, B))^{\mathrm{op}} \to (\mathbf{Poset}(\mathbf{dbTree}(G, A))^{\mathrm{op}},$$

is an indexed category, in which there is a stable meet semilattice homomorphism

$$h_! = (\mathbf{Poset}(\triangle h))^{\mathrm{op}} : (\mathbf{Poset}(\mathbf{dbTree}(G, A))^{\mathrm{op}} \to (\mathbf{Poset}(\mathbf{dbTree}(G, B))^{\mathrm{op}}$$

for each path $h : A \to B$ *in* G.

By Grothendieck construction, we have the fibration $\partial_G : \mathrm{g}(\mathbf{PosetdbTree}(G)) \to G^*$ constructed from the indexed category *PosetdbTree* $: (G^*)^{\mathrm{op}} \to \mathbf{msLat}$. More precisely, $\mathrm{g}(\mathbf{PosetdbTree}(G))$ is the category with the following data:

objects: $(A, ((T, b_T), \nu_T))$,

where $A \in V(T)$ and $((T, b_T), \nu_T) \in \mathbf{PosetdbTree}(G, A)$;

maps: a map from $(A, ((T, b_T), \nu_T))$ to $(B, ((S, b_S), \nu_S))$ is a path $h : A \to B$ in G such that

$$(A, ((T, b_T), \nu_T)) \leq h^*(B, ((S, b_S), \nu_S)) \text{ in } (\mathbf{PosetdbTree}(G, A))^{\mathrm{op}};$$

composition and **identities** are formed as in G^*.

The forgetful functor $\partial_G : \mathrm{g}(\mathbf{PosetdbTree}(G)) \to G^*$ turns out to be what we wanted as showed by the following lemma.

Lemma 5.2.11 *The fibration* $\partial_G : \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to G^*$ *is a range stable meet semilattice fibration.*

PROOF: For any $A \in V(G)$, we have

$$\partial_G^{-1}(A) = \{(A, ((T, b_T), \nu_T)) \mid ((T, b_T), \nu_T) \in (\mathbf{Poset}(\mathbf{dbTree}(G, A))^{\mathrm{op}}\}.$$

Since $(\mathbf{Poset}(\mathbf{dbTree}(G, A))^{\mathrm{op}}$ is a meet semilattice, $\partial_G^{-1}(A)$ is also a meet semilattice with the order and meet of $(\mathbf{Poset}(\mathbf{dbTree}(G, A))^{\mathrm{op}}$. For any path $h : A \to B$ in G, we have stable meet semilattice homomorphisms $h^* : \partial_G^{-1}(B) \to \partial_G^{-1}(A)$ sending $(B, ((S, b_S), \nu_S))$ to $(A, \Box h((S, b_S), \nu_S))$ and $h_! : \partial_G^{-1}(A) \to \partial_G^{-1}(B)$ sending $(A, ((T, b_T), \nu_T))$ to $(B, \triangle h((T, b_T), \nu_T))$. Hence $\partial_G : \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to G^*$ is a stable meet semilattice fibration. Now it suffices to show that ∂_G satisfies [**rsF**.**1**], [**rsF**.**2**], and [**rsF**.**3**]. Let $(A, ((T, b_T), \nu_T)) \in \partial_G^{-1}(A)$ and $(B, ((S, b_S), \nu_S)) \in \partial_G^{-1}(B)$.

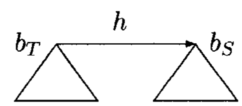
[**rsF**.**1**] Compute

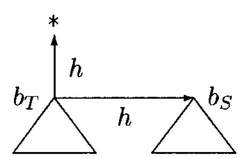
$$\begin{aligned}
& (h_!(A, ((T, b_T), \nu_T))) \wedge (B, ((S, b_S), \nu_S)) \\
= \;& (B, Det((T, b_T)^-, \hat{h}\nu_T^-)) \wedge (B, ((S, b_S), \nu_S)) \\
= \;& (B, Det((T, b_T)^-, \hat{h}\nu_T^-) + ((S, b_S), \nu_S)),
\end{aligned}$$

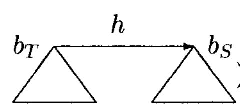
and

$$
\begin{aligned}
& h_!((A,((T,b_T),\nu_T)) \wedge h^*(B,((S,b_S),\nu_S))) \\
= \; & h_!((A,((T,b_T),\nu_T)) \wedge (A,((S,b_S)^+,\check{h}\nu_S^+))) \\
= \; & h_!(A,((T,b_T),\nu_T) + ((S,b_S)^+,\check{h}\nu_S^+)) \\
= \; & (B, h_!(((T,b_T),\nu_T) + ((S,b_S)^+,\check{h}\nu_S^+)))
\end{aligned}
$$

as displayed graphically:

$(h_!(A,((T,b_T),\nu_T))) \wedge (B,((S,b_S),\nu_S)) \; = \; Det($  $)$

$h_!((A,((T,b_T),\nu_T)) \wedge h^*(B,((S,b_S),\nu_S))) \; = Det($  $)$

$= Det($  $)$

Hence, there is a map

$$h_!((T,b_T),\nu_T) + ((S,b_S)^+,\check{h}\nu_S^+) \to Det((T,b_T)^-,\hat{h}\nu_T^-) + ((S,b_S),\nu_S),$$

and therefore

$$
\begin{aligned}
& (h_!(A, ((T, b_T), \nu_T))) \wedge (B, ((S, b_S), \nu_S)) \\
\leq \;& h_!((A, ((T, b_T), \nu_T)) \wedge h^*(B, ((S, b_S), \nu_S)).
\end{aligned}
$$

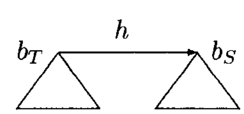
[rsF.2] Compute

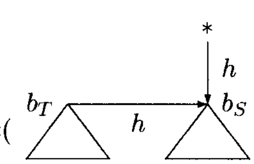
$$
\begin{aligned}
& (A, ((T, b_T), \nu_T)) \wedge (h^*(B, ((S, b_S), \nu_S))) \\
= \;& (A, ((T, b_T), \nu_T)) \wedge (A, ((S, b_S)^+, \check{h}\nu_S^+)) \\
= \;& (A, ((T, b_T), \nu_T) + ((S, b_S)^+, \check{h}\nu_S^+)),
\end{aligned}
$$

and

$$
\begin{aligned}
& h^*(h_!(A, ((T, b_T), \nu_T)) \wedge (B, ((S, b_S), \nu_S))) \\
= \;& h^*(B, Det((T, b_T)^-, \hat{h}\nu_T^-) + ((S, b_S), \nu_S)) \\
= \;& (A, h^*(Det((T, b_T)^-, \hat{h}\nu_T^-) + ((S, b_S), \nu_S)))
\end{aligned}
$$

as displayed graphically:

$(A, ((T, b_T), \nu_T)) \wedge (h^*(B, ((S, b_S), \nu_S))) \;=\; Det($  $)$

$h^*(h_!(A, ((T, b_T), \nu_T)) \wedge (B, ((S, b_S), \nu_S))) \;= Det($  $)$

Hence, there is a map

$$h^*(Det((T,b_T)^-, \hat{h}\nu_T^-) + ((S,b_S),\nu_S)) \to ((T,b_T),\nu_T) + ((S,b_S)^+, \check{h}\nu_S^+).$$

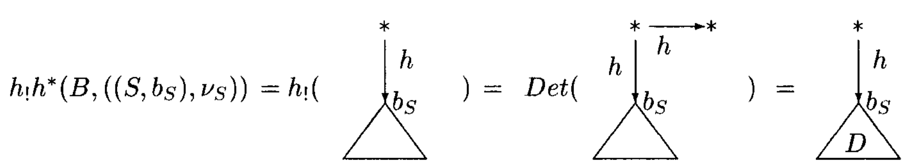
Therefore,

$$\begin{aligned} & (A, ((T,b_T),\nu_T)) \wedge (h^*(B, ((S,b_S),\nu_S))) \\ \leq \; & h^*(h_!(A, ((T,b_T),\nu_T)) \wedge (B, ((S,b_S),\nu_S))). \end{aligned}$$

[rsF.3] Note that

$$\begin{aligned} h_! h^*(B, ((S,b_S),\nu_S)) &= h_!(A, h^*((S,b_S),\nu_S)) \\ &= h_!(A, ((S,b_S)^+, \check{h}\nu_S^+)) \\ &= (B, Det(((S,b_S)^+)^-, \hat{h}(\check{h}\nu_S^+)^-)) \end{aligned}$$

as displayed graphically:



Clearly, there is a map $((S,b_S),\nu_S) \to (Det((S,b_S)^+)^-, \hat{h}(\check{h}\nu_S^+)^-)$ and therefore

$$h_! h^*(B, ((S,b_S),\nu_S)) \leq (B, ((S,b_S),\nu_S)).$$

Hence $\partial_G : \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to G^*$ is a range stable meet semilattice fibration.

□

5.3 The Free Range Restriction Categories over Directed Graphs

In this section, we prove that $\partial_G : \mathrm{g}(\mathbf{PostdbTree}(G)) \to G^*$ constructed in the last section is the free range stable meet semilattice fibration and produces the free range restriction category over a given directed graph G. We also provide some examples of free range restriction categories over directed graphs.

5.3.1 ∂_G is Free

If $\alpha : G \to H$ is a map in **Graph**, then we have a functor $\alpha^* : G^* \to H^*$ and a functor

$$\alpha_* : \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(H)))$$

given by sending

$$h : (A, ((T, b_T), \nu_T)) \to (B, ((S, b_S), \nu_S))$$

to

$$\alpha^*(h) : (\alpha(A), ((T, b_T), \alpha\nu_T)) \to (\alpha(B), ((S, b_S), \alpha\nu_S)).$$

Clearly,

$$\begin{array}{ccc} \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) & \xrightarrow{\alpha_*} & \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(H))) \\ {\scriptstyle \partial_G}\downarrow & & \downarrow{\scriptstyle \partial_H} \\ G^* & \xrightarrow{\alpha^*} & H^* \end{array}$$

commutes. For any path $h : A \to B$ in G, and

$$(A, ((T, b_T), \nu_T)) \in \partial_G^{-1}(A), (S, ((T, b_S), \nu_S)), (A, ((R, b_R), \nu_R)) \in \partial_G^{-1}(B),$$

we have

[sfM.1] $\alpha_*(\top_{\partial_G^{-1}(A)}) = \alpha_*(A, ((A, A), 1_A)) = (\alpha(A), ((A, A), \alpha 1_A)) = \top_{\partial_H^{-1}(\alpha(A))}$,

[sfM.2]

$$\begin{aligned} & \alpha_*((B, ((S, b_S), \nu_S)) \wedge (B, ((R, b_R), \nu_R))) \\ = \; & \alpha_*((B, (Det(S + R, b_{S+R}), \nu_{S+R}))) \\ = \; & (\alpha(B), (Det(S + R, b_{S+R}), \alpha\nu_{S+R})) \\ = \; & (\alpha(B), ((S, b_S), \alpha\nu_S)) \wedge (\alpha(B), ((R, b_R), \alpha\nu_R)) \\ = \; & \alpha_*(B, ((S, b_S), \nu_S)) \wedge \alpha_*(B, ((R, b_R), \nu_R)), \end{aligned}$$

[sfM.3]

$$\begin{aligned} \alpha_*(h^*(B, ((S, b_S), \nu_S))) &= \alpha_*(A, (h^*((S, b_S), \nu_S))) \\ &= \alpha_*(A, ((S, b_S)^+, \check{h}\nu_S^+)) \\ &= (\alpha(A), ((S, b_S)^+, \alpha\check{h}\nu_S^+)) \\ &= (\alpha^*(h))^*(\alpha(B), ((S, b_S), \alpha\nu_S)) \\ &= (\alpha^*(h))^*(\alpha_*(B, ((S, b_S), \nu_S))), \end{aligned}$$

[rsfM.1]

$$\begin{aligned}
\alpha_*(h_!(A,((T,b_T),\nu_T))) &= \alpha_*(B,(h_!((T,b_T),\nu_T))) \\
&= \alpha_*(B,((S,b_S)^-,\hat{h}\nu_S^-)) \\
&= (\alpha(B),((S,b_S)^-,\alpha\hat{h}\nu_S^-)) \\
&= (\alpha^*(h))^*(\alpha(A),((T,b_T),\alpha\nu_T)) \\
&= (\alpha^*(h))^*(\alpha_*(A,((T,b_T),\nu_T))).
\end{aligned}$$

Hence $(\alpha^*, \alpha_*) : \partial_G \to \partial_H$ is a map in $\mathbf{rsFib}_0$. So we have:

Lemma 5.3.1 *Let $\alpha : G \to H$ be a path in G. Then $(\alpha^*, \alpha_*) : \partial_G \to \partial_H$ is a map in $\mathbf{rsFib}_0$.*

For any map $\alpha : G \to H$ in **Graph**, by Lemma 5.3.1 there is a map (α^*, α_*) : $\partial_G \to \partial_H$ in $\mathbf{rsFib}_0$. Hence, we have a functor F_{rf} : **Graph** $\to \mathbf{rsFib}_0$, which sends $\alpha : G \to H$ to $(\alpha^*, \alpha_*) : \partial_G \to \partial_H$. In the other direction, there is a functor $U_{rf} : \mathbf{rsFib}_0 \to$ **Graph**, which sends $(F, F') : \delta_{\mathbf{X}} \to \delta_{\mathbf{Y}}$ to $F : \mathbf{X} \to \mathbf{Y}$ forgetting composition and identities.

In order to prove that $F_{rf} \dashv U_{rf}$, we first develop a method to write a based tree inductively. For any based tree (T, b_T), let

$$\mathcal{D}^T_{b_T} = \{e \in V(T) \mid \partial_0(e) = b_T\}$$

and

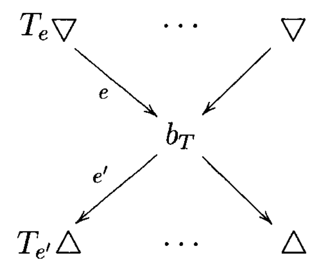
$$\mathcal{R}^T_{b_T} = \{e \in V(T) \mid \partial_1(e) = b_T\}.$$

Write (T, b_{T_e}) as

$$(T, b_T) = (+_{e \in \mathcal{D}^T_{b_T}} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}^T_{b_T}} e^*(T_e, b_{T_e})),$$

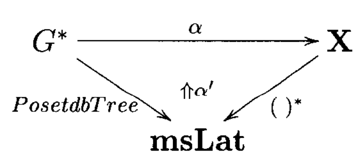
where (T_e, b_{T_e}) are based trees. We can inductively repeat the above process in each based tree (T_e, b_{T_e}) so that we can write (T, b_T) as the coproducts of some initial based trees $(*, *)$ by using the operations $(\ )^*$ and $(\ )_!$ as displayed graphically as follows:

$(T, b_T) \quad =$



Lemma 5.3.2 *For any range stable meet semilattice fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *and any direct graph map* $\alpha : G \to U_{rf}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$*, there is a unique functor* $\tilde{\alpha}$: $\mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to \tilde{\mathbf{X}}$ *such that* $(\alpha, \tilde{\alpha}) : \partial_G \to \delta_{\mathbf{X}}$ *is a map in* $\mathbf{rsFib}_0$.

Proof: Recall that in order to define a map $(\alpha, \tilde{\alpha}) : \partial_G \to \delta_{\mathbf{X}}$ in $\mathbf{rsFib}_0$, we need to define a natural transformation $\alpha' : PosetdbTree \to (\ )^*\alpha$, where $(\ )^*$ is the indexed category corresponding to the fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$:



In order to define a meet preserving functor $\alpha'_A : \partial_G^{-1}(A) \to \delta_{\mathbf{X}}^{-1}(\alpha(A))$, for each $(A, ((T, b_T), \nu_T)) \in \partial_G^{-1}(A)$, we write

$$(T, b_T) = (+_{e \in \mathcal{D}_{b_T}^T} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}_{b_T}^T} e^*(T_e, b_{T_e})).$$

Define $\alpha'_A(A, ((T, b_T), \nu_T)) = \alpha'_A(T, b_T)$, which is given inductively as follows:

$$\begin{aligned}
\alpha'_A(*, *) &= \top_{\delta_{\mathbf{X}}^{-1}(\alpha\nu_T(*))} \\
\alpha'_A(e^*(*, *)) &= (\alpha(\nu_T(e)))^* \top_{\delta_{\mathbf{X}}^{-1}(\alpha\nu_T(*))} \\
\alpha'_A(e_!(*, *)) &= (\alpha(\nu_T(e)))_! \top_{\delta_{\mathbf{X}}^{-1}(\alpha\nu_T(*))} \\
\alpha'_A((+_{e \in \mathcal{D}_{b_T}^T} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}_{b_T}^T} e^*(T_e, b_{T_e}))) &= (\wedge_{e \in \mathcal{D}_{b_T}^T} (\alpha\nu_T(e))_! \alpha'_A(T_e, b_{T_e})) \\
&\quad \wedge (\wedge_{e \in \mathcal{R}_{b_T}^T} (\alpha\nu_T(e))^* \alpha'_A(T_e, b_{T_e})).
\end{aligned}$$

If there is a map $\kappa : (T, b_T) \to (S, b_S)$, then $\alpha'_A(T, b_T) \geq \alpha'_A(S, b_S)$. In fact, if $(T, b_T) = (*, *)$, then, clearly,

$$\alpha'_A(T, b_T) = \alpha'_A(*, *) = \top_{\delta_{\mathbf{X}}^{-1}(\alpha\nu_T(*))} \geq \alpha'_A(S, b_S).$$

Otherwise, we write (T, b_T) as

$$(T, b_T) = (+_{e \in \mathcal{D}_{b_T}^T} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}_{b_T}^T} e^*(T_e, b_{T_e})).$$

Then for each based tree (T_e, b_{T_e}), there is a based tree $(S_{i(e)}, b_{T_{i(e)}})$ such that there is a map $\kappa_e : (T_e, b_{T_e}) \to (S_{i(e)}, b_{T_{i(e)}})$. By induction, $\alpha'_A(T_e, b_{T_e}) \geq \alpha'_A(S_{i(e)}, b_{T_{i(e)}})$.

Therefore,

$$\begin{aligned}
\alpha'_A(T, b_T) &= (\wedge_{e\in\mathcal{D}^T_{b_T}}(\alpha\nu_T(e))_!\alpha'_A(T_e, b_{T_e})) \wedge (\wedge_{e\in\mathcal{R}^T_{b_T}}(\alpha\nu_T(e))^*\alpha'_A(T_e, b_{T_e})) \\
&\geq (\wedge_{i(e)\in\mathcal{D}^S_{b_S}}(\alpha\nu_S(i(e)))_!\alpha'_A(S_{i(e)}, b_{S_{i(e)}})) \\
&\quad \wedge(\wedge_{i(e)\in\mathcal{R}^S_{b_S}}(\alpha\nu_S(i(e)))^*\alpha'_A(S_{i(e)}, b_{S_{i(e)}})) \\
&\geq \alpha'_A(S, b_S).
\end{aligned}$$

It is easy to check that for any map $h : A \to B$ in G^*,

$$\begin{array}{ccc}
\partial_G^{-1}(B) & \xrightarrow{\alpha'_B} & \delta_{\mathbf{X}}^{-1}(B) \\
{\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle (\alpha(h))^*} \\
\partial_G^{-1}(A) & \xrightarrow{\alpha'_A} & \delta_{\mathbf{X}}^{-1}(A)
\end{array}$$

commutes. Hence $\alpha' : PosetdbTree \to (\ )^*\alpha$ is a natural transformation. Note that

$$\alpha' : PosetdbTree \to (\ )^*\alpha$$

yields a functor $\tilde{\alpha} : \mathrm{g}(\mathbf{PostdbTree}(G)) \to \tilde{\mathbf{X}}$ which sends

$$h : (A, ((T, b_T), \nu_T)) \to (B, ((S, b_S), \nu_S))$$

to

$$\tilde{\alpha}(h) : \alpha'_A(T, b_T) \to \alpha'_B(S, b_S),$$

where $\tilde{\alpha}(h)$ is given by the following commutative diagram in $\tilde{\mathbf{X}}$:

$$\begin{array}{ccc} & \alpha'_A(T, b_T) & \\ \swarrow^{\leq} & & \searrow^{\tilde{\alpha}(h)} \\ (\alpha(h))^*(\alpha'_B(S, b_S)) & \xrightarrow{\vartheta_{\alpha(h)}} & \alpha'_B(S, b_S) \end{array}$$

Now it is routine to check that

$$\begin{array}{ccc} \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) & \xrightarrow{\tilde{\alpha}} & \tilde{\mathbf{X}} \\ {\scriptstyle \partial_G}\downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{X}}} \\ G^* & \xrightarrow[\alpha]{} & \mathbf{X} \end{array}$$

commutes.

For any map $h : A \to B$ in G^*, any $(A, ((T, b_T), \nu_T)) \in \partial_G^{-1}(A)$, and any

$$(B, ((S, b_S), \nu_S)), (B, ((Q, b_Q), \nu_Q)) \in \partial_G^{-1}(B),$$

we have

$$\begin{aligned} \tilde{\alpha}(\top_{\partial_G^{-1}(A)}) &= \tilde{\alpha}(A, ((A, A), 1_A)) \\ &= (\alpha^*(A, [\,], A))^*(\top_{\delta_{\mathbf{X}}^{-1}(\alpha(A))}) \wedge (\alpha^*(A, [\,], A))_!(\top_{\delta_{\mathbf{X}}^{-1}(\alpha(A))}) \\ &= \top_{\delta_{\mathbf{X}}(\alpha(A))}, \end{aligned}$$

and

$$
\begin{aligned}
& \tilde{\alpha}((B,((S,b_S),\nu_S)) \wedge (B,((Q,b_Q),\nu_Q))) \\
= \; & \tilde{\alpha}(B,((S+Q,b_{S+Q}),\nu_{S+Q})) \\
= \; & (\wedge_{e\in\mathcal{D}^{S+Q}_{b_{S+Q}}}(\alpha\nu_{S+Q}(e))_!\alpha'_A((S+Q)_e,b_{(S+Q)_e})) \\
& \quad \wedge(\wedge_{e\in\mathcal{R}^{S+Q}_{b_{S+Q}}}(\alpha\nu_{S+Q}(e))^*\alpha'_A((S+Q)_e,b_{(S+Q)_e})) \\
= \; & \tilde{\alpha}(B,((S,b_S),\nu_S)) \wedge \tilde{\alpha}(B,((Q,b_Q),\nu_Q)),
\end{aligned}
$$

since

$$
\mathcal{D}^{S+Q}_{b_{S+Q}} = \mathcal{D}^{S}_{b_S} \cup \mathcal{D}^{Q}_{b_Q},
$$

$$
\mathcal{R}^{S+Q}_{b_{S+Q}} = \mathcal{R}^{S}_{b_S} \cup \mathcal{R}^{Q}_{b_Q},
$$

$$
\begin{aligned}
\tilde{\alpha}(h^*(B,((S,b_S),\nu_S))) &= \tilde{\alpha}(A,h^*((S,b_S),\nu_S))) \\
&= (\alpha\nu_S(+))^*\tilde{\alpha}(S,b_S) \\
&= (\alpha(h))^*(\tilde{\alpha}(B,((S,b_S),\nu_S))),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\alpha}(h_!(A,((T,b_T),\nu_T))) &= \tilde{\alpha}(A,h_!((T,b_T),\nu_T)) \\
&= (\alpha\nu_T(-))_!\tilde{\alpha}(T,b_T) \\
&= (\alpha(h))_!(\tilde{\alpha}(A,((T,b_T),\nu_T))).
\end{aligned}
$$

Hence $(\alpha, \tilde{\alpha}) : \partial_G \to \delta_{\mathbf{X}}$ satisfies conditions [**sfM.1**], [**sfM.2**], [**sfM.3**], and [**rsfM.1**] and therefore $(\alpha, \tilde{\alpha})$ is also a map in $\mathbf{rsFib}_0$.

Assume that $\tilde{\alpha}' : \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) \to \tilde{\mathbf{X}}$ is also a functor such that

$$(\alpha, \tilde{\alpha}') : \partial_G \to \delta_{\mathbf{X}}$$

is a map in $\mathbf{rsFib}_0$. Let $h : (A, ((T, b_T), \nu_T)) \to (B, ((S, b_S), \nu_S))$ be a map in $\mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G)))$. Then $(A, ((T, b_T), \nu_T)) \leq h^*(B, ((S, b_S), \nu_S))$. By [**sfM.1**],

$$\tilde{\alpha}'(A, ((A, A), 1)) = \tilde{\alpha}'(\top_{\partial_G^{-1}(A)}) = \top_{\delta_{\mathbf{X}}^{-1}(\alpha(A))} = \tilde{\alpha}(A, ((A, A), 1)).$$

We write

$$(T, b_T) = (+_{e \in \mathcal{D}^T_{b_T}} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}^T_{b_T}} e^*(T_e, b_{T_e})).$$

Assume that $\tilde{\alpha}'(T_e, b_{T_e}) = \tilde{\alpha}(T_e, b_{T_e})$. Since $(\alpha, \tilde{\alpha}')$ is a map in $\mathbf{rsFib}_0$, by [**sfM.2**], [**sfM.2**], and [**rsfM.1**], we have

$$\begin{aligned}
\tilde{\alpha}'(A, ((T, b_T), \nu_T)) &= \tilde{\alpha}'((+_{e \in \mathcal{D}^T_{b_T}} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}^T_{b_T}} e^*(T_e, b_{T_e}))) \\
&= (\wedge_{e \in \mathcal{D}^T_{b_T}} \tilde{\alpha}'(e_!(T_e, b_{T_e}))) \wedge (\wedge_{e \in \mathcal{R}^T_{b_T}} \tilde{\alpha}'(e^*(T_e, b_{T_e}))) \\
&= (\wedge_{e \in \mathcal{D}^T_{b_T}} (\alpha\nu_T(e))_! \tilde{\alpha}'(T_e, b_{T_e})) \wedge (\wedge_{e \in \mathcal{R}^T_{b_T}} (\alpha\nu_T(e))^* \tilde{\alpha}'(T_e, b_{T_e})) \\
&= \tilde{\alpha}((+_{e \in \mathcal{D}^T_{b_T}} e_!(T_e, b_{T_e})) + (+_{e \in \mathcal{R}^T_{b_T}} e^*(T_e, b_{T_e}))) \\
&= \tilde{\alpha}(A, ((T, b_T), \nu_T)).
\end{aligned}$$

Similarly, $\tilde{\alpha}'(B, ((S, b_S), \nu_S)) = \tilde{\alpha}(B, ((S, b_S), \nu_S))$.

For any map $h : (A, ((T, b_T), \nu_T)) \to (B, ((S, b_S), \nu_S))$ in $\mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G)))$, since

$$\begin{array}{ccc} \mathrm{g}(\mathbf{Poset}(\mathbf{dbTree}(G))) & \xrightarrow{\tilde{\alpha}'} & \tilde{\mathbf{X}} \\ {\scriptstyle \partial_G}\downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{X}}} \\ G^* & \xrightarrow[\alpha]{} & \mathbf{X} \end{array}$$

is commutative,

$$\delta_{\mathbf{X}} \tilde{\alpha}'(h) = \alpha \partial_G(h) = \alpha(h).$$

Hence, there is a unique map $x : \tilde{\alpha}(A, ((T, b_T), \nu_T)) \to \tilde{\alpha}'(A, h^*((S, b_S), \nu_S))$ in $\delta_{\mathbf{X}}^{-1}(\alpha(A))$ such that $\delta_{\mathbf{X}}(x) = 1_{\alpha(A)}$ and $\vartheta_{\alpha(h)} x = \tilde{\alpha}'(h)$:

$$\begin{array}{ccccc} & & \tilde{\alpha}(A, ((T, b_T), \nu_T)) & & \\ & \swarrow{\scriptstyle \exists! x} & & \searrow{\scriptstyle \tilde{\alpha}'(h)} & \\ (\alpha(h))^* \tilde{\alpha}(B, ((S, b_S), \nu_S)) & & \xrightarrow{\vartheta_{\alpha(h)}} & & \tilde{\alpha}(B, ((S, b_S), \nu_S)) \end{array}$$

$$\downarrow{\scriptstyle \delta_{\mathbf{X}}}$$

$$\begin{array}{ccccc} & & \alpha(A) & & \\ & \swarrow{\scriptstyle 1_{\alpha(A)}} & & \searrow{\scriptstyle \alpha(h)} & \\ \alpha(A) & & \xrightarrow{\alpha(h)} & & \alpha(B) \end{array}$$

Since $\delta_{\mathbf{X}}$ is a stable meet semilattice fibration, $x = \leq$. Hence, by the definition of $\tilde{\alpha}$,

$$\tilde{\alpha}'(h) = \vartheta_{\alpha(h)} x = \vartheta_{\alpha(h)} \leq = \tilde{\alpha}(h),$$

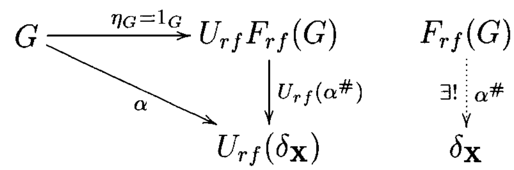
and therefore the uniqueness of $\tilde{\alpha}$ follows. $\square$

Theorem 5.3.3 *There is an adjunction:*

$$\mathbf{rsFib}_0 \underset{U_{rf}}{\overset{F_{rf}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Graph}$$

with the identity unit $\eta_G = 1_G$.

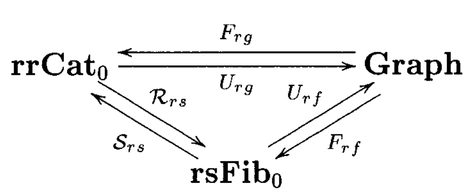
PROOF: For any direct graph G, clearly $U_{rf}F_{rf}(G) = G$. $\eta_G = 1_G : G \to U_{rf}F_{rf}(G)$ turns out to be the unit of $F_{rf} \dashv U_{rf}$. In fact, for any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ and any direct graph map $\alpha : G \to U_{rf}(\delta_{\mathbf{X}})$, by Lemma 5.3.2, we have a unique map $\alpha^{\#} = (\alpha, \tilde{\alpha}) : \partial_G \to \delta_{\mathbf{X}}$ in $\mathbf{rsFib}_0$ such that



commutes. Hence $F_{rf} \dashv U_{rf}$ with the unit $\eta_G = 1_G$. □

Theorem 4.1.11 states that there is an adjunction: $\mathbf{rrCat}_0 \underset{\mathcal{R}_{rs}}{\overset{\mathcal{S}_{rs}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{rsFib}_0$. Hence we have the functor $U_{rg} = U_{rf}\mathcal{R}_{rs} : \mathbf{rrCat}_0 \to \mathbf{Graph}$ and the functor $F_{rg} = \mathcal{S}_{rs}F_{rf}$: $\mathbf{Graph} \to \mathbf{rrCat}_0$. By the adjointness of $\mathcal{S}_{rs} \dashv \mathcal{R}_{rs}$ and $F_{rf} \dashv U_{rf}$, it is easy to see that $F_{rg} \dashv U_{rg}$. So we have:

Theorem 5.3.4 *There is a commutative adjunction diagram:*



So, for any directed graph G, $F_{rf}(G)$ is the *free range stable meet semilattice fibration over* G and $F_{rg}(G)$ is the *free range restriction category over* G.

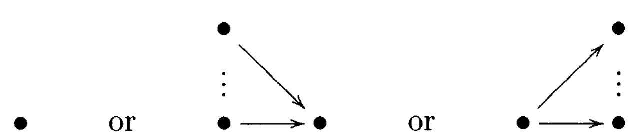
5.3.2 The Free Range Restriction Category over an Arrow

Let G be the directed graph displayed by $A \xrightarrow{f} B$. Then G^* is the path category with

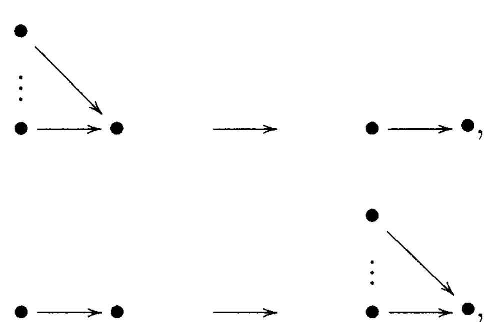
objects: A, B;

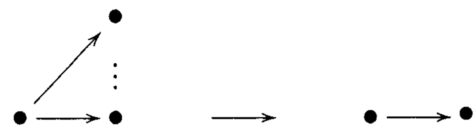
maps: $1_A = (A, [\,], A) : A \to A, 1_B = (B, [\,], B) : B \to B, (A, [f], B) : A \to B$.

If $((T, b_T), \nu_T) \in \mathbf{dbTree}(G, *)$, then there are no paths $[m_1, m_2]$ in T since m_1 and m_2 must satisfy $\nu_T(m_1) = \nu_T(m_2) = f$. Obviously, $\mathbf{dbTree}(G, B) = \mathbf{dbTree}(G, A)$ is given by the set of all possible trees in G displayed as

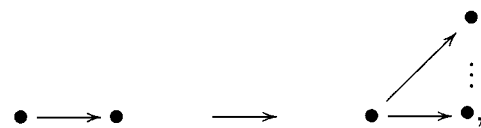


with properly chosen bases (so that they are in $\mathbf{dbTree}(G, A)$ or $\mathbf{dbTree}(G, B)$). Since there are obvious maps:





and



we have

$$\mathbf{Poset}(\mathbf{dbTree}(G, B)) = \{((*, *), \iota), ((\bullet \longrightarrow *, *), 1)\}$$

and

$$\mathbf{Poset}(\mathbf{dbTree}(G, A)) = \{((*, *), \iota), ((* \longrightarrow \bullet, *), 1)\}.$$

Hence, we have the indexed category $PosetdbTree : (G^*)^{\mathrm{op}} \to \mathbf{msLat}$, and so by Grothendieck construction, the free range stable meet semilattice fibration

$$\partial_G : \mathrm{g}(\mathbf{PosetdbTree}(G)) \to G^*,$$

which is the forgetful functor, where $\mathrm{g}(\mathbf{PosetdbTree}(G))$ is the category with

objects: $(A, ((*, *), \iota)), (A, ((* \longrightarrow \bullet, *), 1)), (B, ((*, *), \iota)),$
$(B, ((\bullet \longrightarrow *, *), 1))$;

maps: a map from $(t, ((T, b_T), \nu_T))$ to $(s, ((S, b_S), \nu_S))$ is a map $h : t \to s$ in G^* such that

$$(t, ((T, b_T), \nu_T)) \leq h^*(s, ((S, b_S), \nu_S))$$

in $(\mathbf{PosetdbTree}(G,*))^{\mathrm{op}}$, where $t, s \in \{A, B\}$.

By Theorem 5.3.4, $F_{rg}(G) = \mathcal{S}_{rs}(\partial_G)$. Explicitly, $F_{rg}(G)$ is the category with

objects: the same as G^*, namely, A,B;

maps: $(h, \sigma) : t \to s$, where $h : t \to s$ is a map in G^* and $\sigma \in \partial_G^{-1}(t)$ is such that

$$\sigma \leq h^*(\top_{\partial_G^{-1}(s)}),$$

where $t, s \in \{A, B\}$;

composition: $(g, \sigma_2)(h, \sigma_1) = (gh, \sigma_1 \wedge h^*(\sigma_2))$;

identities: $1_A = (1_A, \top_{\partial_G^{-1}(A)}) = (1, ((*, *), \iota)), 1_B = (1_B, \top_{\partial_G^{-1}(B)}) = (1, ((*, *), \iota))$.

Now, let's look at the maps of $F_{rg}(G)$. It is easy to check that

$$(1_A, (A, ((*, *), \iota))) : A \to A,$$

$$(1_A, (A, ((* \longrightarrow \bullet, *), 1))) : A \to A,$$

$$(1_B, (B, ((*, *), \iota))) : B \to B,$$

$$(1_B, (B, ((\bullet \longrightarrow *, *), 1))) : B \to B,$$

and

$$((A, [f], B), (A, ((* \longrightarrow \bullet, *), 1))) : A \to B$$

are maps in $F_{rg}(G)$. But

$$((A,[f],B),(A,((*,*),\iota))) : A \to B$$

is not a map in $F_{rg}(G)$ since

$$(A,((*,*),\iota)) \not\leq (A,[f],B)^*(\top_{\partial_G^{-1}(B)}) = (A,[f],B)^*(B,((*,*),\iota)).$$

The range and restriction structures are given by

$$\overline{(1_A,(A,((*,*),\iota)))} = (1_A,(\widehat{A,((*,*),\iota)})) = (1_A,(A,((*,*),\iota))),$$

$$\begin{aligned}\overline{(1_A,(A,((*\longrightarrow\bullet\,,*),1)))} &= (1_A,(A,((\widehat{*\longrightarrow\bullet}\,,*),1))) \\ &= (1_A,(A,((*\longrightarrow\bullet\,,*),1))),\end{aligned}$$

$$\overline{(1_B,(B,((*,*),\iota)))} = (1_B,(\widehat{B,((*,*),\iota)})) = (1_B,(B,((*,*),\iota))),$$

$$\begin{aligned}\overline{(1_B,(B,((\bullet\longrightarrow*\,,*),1)))} &= (1_B,(B,((\widehat{\bullet\longrightarrow*}\,,*),1))) \\ &= (1_B,(B,((\bullet\longrightarrow*\,,*),1))),\end{aligned}$$

$$\overline{((A,[f],B),(A,((*\longrightarrow\bullet\,,*),1)))} = (1_A,(A,((*\longrightarrow\bullet\,,*),1))),$$

and

$$((A,[f],B),(\widehat{A,((*\longrightarrow\bullet}\,,*),1))) = (1_B,(B,((\bullet\longrightarrow*\,,*),1))).$$

Write

$$((A,[f],B),(A,((\,* \longrightarrow \bullet\,,*),1))) = f,$$

$$(1_A,(A,((*,*),\iota))) = 1_A,$$

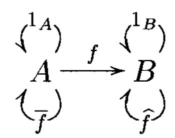
$$(1_A,(A,((\,* \longrightarrow \bullet\,,*),1))) = \overline{f},$$

$$(1_B,(B,((*,*),\iota))) = 1_B,$$

and

$$(1_B,(B,((\,\bullet \longrightarrow *\,,*),1))) = \widehat{f}.$$

Then $F_{rg}(G)$ is the free range restriction category displayed by



with the obvious composition.

5.3.3 The Free Range Restriction Category over a Single Endoarrow

Let H be the directed graph displayed by $\bullet \circlearrowright f$. Then H^* is the category with $\bullet$ as its unique object, with all possible trees displayed by $\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \;\cdots\; \bullet \xrightarrow{f} \bullet$ as maps, and with concatenation as the composition. If we denote the tree

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \;\cdots\; \bullet \xrightarrow{f} \bullet$$

with n edges by n, then clearly, H^* is the category $(\bullet, \mathbb{N})$ with the composition given by $+$. We consider the free range restriction category over $(\bullet, \mathbb{N})$, which is $F_{rg}(H)$.

We denote $\mathsf{Total}(\mathsf{Split}_E(F_{rg}(H)))$ by $rr\mathbb{N}$ and take a quick look at its structure which is quite interesting.

Let $(T,*)$ be an arbitrary based tree, and let ν_T send each vertex of T to $\bullet$ and each edge of T to f. Then clearly, $((T,*),\nu_T) \in \mathbf{bTree}(H,\bullet)$. If $((T,*),\nu_T) \in \mathbf{Poset}(\mathbf{dbTree})(H,\bullet)$, then, by an easy induction on the number of edges in T, $((T,*),\nu_T)$ can be displayed as

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & F_{1,1} & & F_{2,1} & & \cdots & & F_{s,1} \\
 & & & & & & & & \downarrow f & & \downarrow f & & & & \downarrow f \\
 & & & & & & & & \cdots & & \cdots & & \cdots & & \cdots \\
 & & & & & & & & \downarrow f & & \downarrow f & & & & \downarrow f \\
 & & & & & & & & F_{1,k_1} & & F_{2,k_2} & & \cdots & & F_{s,k_s} \\
 & & & & & & & & \downarrow f & & \downarrow f & & & & \downarrow f \\
I_1 & \xrightarrow{f} & \cdots & \xrightarrow{f} & I_m & \xrightarrow{f} & * & \xrightarrow{f} & O_1 & \xrightarrow{f} & O_2 & \xrightarrow{f} & \cdots & \xrightarrow{f} & O_s
\end{array}
$$

where the integers $m, s, k_1, \cdots k_s \geq 0$ satisfy

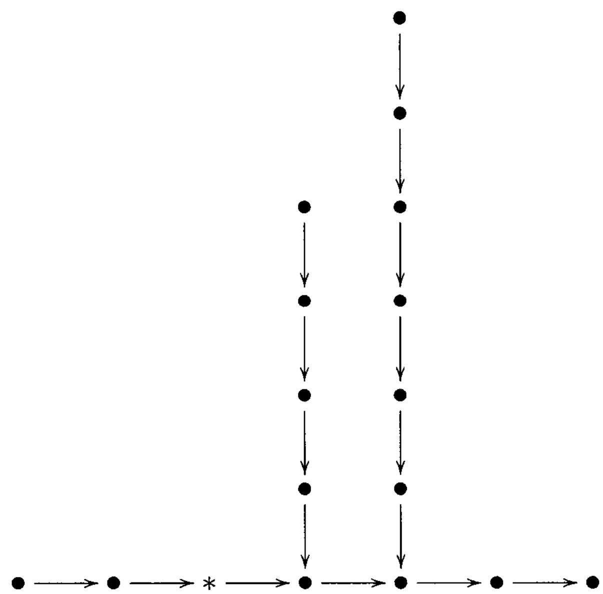
$$k_i > m + i + 1 \text{ or } k_i = 0,$$

$i = 1, \cdots, s$. We denote $((T,*),\nu_T)$ by

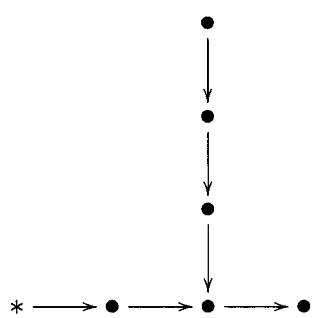
$$(m, [(k_1 - m - 1) \vee 0, (k_2 - m - 2) \vee 0, \cdots, (k_s - m - s) \vee 0]),$$

where $s \vee t$ denotes $\max(s,t)$.

For example, $(2,[1,2,0,0])$ represents the following deterministic based tree:



while $(0,[0,1,0])$ and $(0,[\,])$ represent



and $*$,

respectively. Obviously,

$$\mathbf{Poset}(\mathbf{dbTree}(H,\bullet)) = \{(m,[k_1,\cdots,k_s]) \mid m,s,k_1,\cdots,k_s \geq 0 \text{ are integers}\}.$$

Hence, we have the indexed category $PosetdbTree : (H^*)^{\mathrm{op}} \to \mathbf{msLat}$, and so by

Grothendieck construction, the free range stable meet semilattice fibration

$$\partial_G : \mathrm{g}(\mathbf{PosetdbTree}(H)) \to H^*,$$

which is the forgetful functor, where $\mathrm{g}(\mathbf{PosetdbTree}(H))$ is the category with

objects: $(\bullet, (m, [k_1, \cdots, k_s]))$, where $m, s, k_1, \cdots, k_s \geq 0$ are integers;

maps: a map from $(\bullet, (m, [k_1, \cdots, k_s]))$ to $(\bullet, (n, [l_1, \cdots, l_t]))$ is a $p \in \mathbb{N}$ such that there is a map from $p^*(n, [l_1, \cdots, l_t])$ to $(m, [k_1, \cdots, k_s])$ in $\mathbf{PosetdbTree}(H, \bullet)$;

identities: 0;

composition: $+$.

By Theorem 5.3.4, $F_{rg}(H) = \mathcal{S}_{rs}(\partial_H)$. Explicitly, $F_{rg}(H)$ is the range restriction category with

objects: $\bullet$,

maps: $(p, (m, [k_1, \cdots, k_s])) : \bullet \to \bullet$, where $p \in \mathbb{N}$ and

$$(m, [k_1, \cdots, k_s]) \in \mathbf{Poset}(\mathbf{dbTree}(H, \bullet))$$

with $p \leq s$,

composition:

$$\begin{aligned} & (q, (n, [l_1, \cdots, l_t]))(p, (m, [k_1, \cdots, k_s])) \\ = \; & (q + p, (m, [k_1, \cdots, k_s]) \wedge p^*(n, [l_1, \cdots, l_t])), \end{aligned}$$

identities: $1_\bullet = (0, (0, [\,]))$,

restriction: $\overline{(p, (m, [k_1, \cdots, k_s]))} = (0, (m, [k_1, \cdots, k_s]))$,

range: $(p, (m, \widehat{[k_1, \cdots}, k_s])) = (0, p_!(m, [k_1, \cdots, k_s]))$,

where

$$f_y(x) = \begin{cases} 0 & \text{if } y = 0 \\ x & \text{otherwise} \end{cases}$$

$$(m, [k_1, \cdots, k_s]) \wedge (n, [l_1, \cdots, l_t]) = \begin{cases} (m \vee n, [a_1, \cdots, a_t, b_{t+1}, \cdots, b_s]) & \text{if } s \geq t \\ (m \vee n, [a_1, \cdots, a_s, c_{s+1}, \cdots, c_t]) & \text{otherwise} \end{cases}$$

with

$$a_i = (f_{k_i}(k_i + m + i) \vee f_{l_i}(l_i + n + i) - m \vee n - i) \vee 0,$$

$$b_j = (f_{k_j}(k_j + m + j) - m \vee n) \vee 0,$$

and

$$c_j = (f_{l_j}(l_j + n + j) - m \vee n) \vee 0,$$

$$\begin{aligned} & p^*(m, [k_1, \cdots, k_s]) \\ = \;& (0, [\underbrace{0, \cdots, 0}_{p-1}, (m - p) \vee 0, f_{k_1}(k_1 + m - p) \vee 0, \cdots, f_{k_s}(k_s + m - p) \vee 0]), \end{aligned}$$

$$
\begin{aligned}
& p_!(m, [k_1, \cdots, k_s]) \\
= & \begin{cases} (w, [(f_{k_{p+1}}(k_{p+1} + m + p + 1) - w) \vee 0, \cdots, \\ \quad (f_{k_s}(k_s + m + s + 1) - w) \vee 0]), \text{ if } p < s \\ (0, [\,]), \text{ if } p = s \end{cases}
\end{aligned}
$$

with $w = f_{k_1}(k_1 + m + p) \vee \cdots \vee f_{k_p}(k_p + m + p)$.

By Theorem 2.3.7, $rr\mathbb{N}$ has the following properties:

- admits the $(\mathcal{E}, \mathcal{M})$-factorization system which is pullback stable along $\mathcal{M}$-maps,
- has pullbacks along $\mathcal{M}$-maps,

where

$$\mathcal{E} = \{f : X \to Y \text{ in } rr\mathbb{N} \mid \widehat{f} = 1_Y\}$$

and

$$\mathcal{M} = \{m : X \to Y \text{ in } rr\mathbb{N} | \exists\ r : Y \to X \text{ in } \mathsf{Split}_E(F_{rg}(H)),\ rm = 1_X \text{ and } \overline{mr} = mr\}.$$

The objects of $rr\mathbb{N}$ are restriction idempotents in $F_{rg}(H)$, namely,

$$(0, (m, [k_1, \cdots, k_s]))$$

with integers $m, s, k_1, \cdots, k_s \geq 0$. Clearly, there are countably many objects in this category.

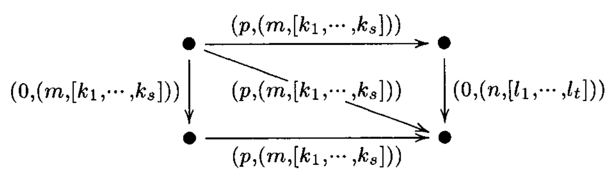
A map $(p, (u, [v_1, \cdots, v_h]))$ from $(0, (m, [k_1, \cdots, k_s]))$ to $(0, (n, [l_1, \cdots, l_t]))$ must satisfy

$$\begin{aligned}(0,(u,[v_1,\cdots,v_h])) &= \overline{(p,(u,[v_1,\cdots,v_h]))}\\ &= 1_{(0,(m,[k_1,\cdots,k_s]))}\\ &= (0,(m,[k_1,\cdots,k_s])).\end{aligned}$$

Hence, it is of the form

$$(p,(m,[k_1,\cdots,k_s])):(0,(m,[k_1,\cdots,k_s]))\to(0,(n,[l_1,\cdots,l_t]))$$

such that



commutes, which means that

$$\begin{aligned}&(m,[k_1,\cdots,k_s])\\ \le\ & p^*(n,[l_1,\cdots,l_t])\\ =\ & (0,[\underbrace{0,\cdots,0}_{p-1},(n-p)\vee 0, f_{l_1}(l_1+n-p)\vee 0,\cdots,f_{l_t}(l_t+n-p)\vee 0])\end{aligned}$$

in $(rr\mathbb{N})^{\mathrm{op}}$. Therefore, a map

$$(p,(u,[v_1,\cdots,v_h])):(0,(m,[k_1,\cdots,k_s]))\to(0,(n,[l_1,\cdots,l_t]))$$

in $\mathsf{Total}(\mathsf{Split}_E(F_{rg}(H)))$ is given by integers $p, u, v_1, \cdots, v_h \geq 0$ satisfying

$$\begin{aligned}
u &= m \\
h &= s \\
u_1 &= m_1 \\
\cdots &\cdots \cdots \\
u_s &= m_s \\
p+t &\leq s \\
(n-p) \vee 0 &\leq k_p \\
f_{l_1}(l_1+n-p) \vee 0 &\leq k_{p+1} \\
\cdots &\cdots \cdots \\
f_{l_t}(l_t+n-p) \vee 0 &\leq k_s
\end{aligned}$$

Clearly, only finitely many possible p's exist or no such p exists, and so

$$\left| rr\mathbb{N}\Big((0,(m,[k_1,\cdots,k_s])),(0,(n,[l_1,\cdots,l_t]))\Big)\right| < +\infty.$$

Also, given $(0,(n,[l_1,\cdots,l_t]))$ and $(p,(u,[v_1,\cdots,v_h]))$, one can easily decide whether or not $(p,(u,[v_1,\cdots,v_h]))$ is a map from $(0,(u,[v_1,\cdots,v_h]))$ to $(0,(n,[l_1,\cdots,l_t]))$. For example, it is easy to see that $(1,(1,[2,1,1])) : (0,(1,[2,1,1])) \to (0,(1,[1,1]))$ is a map in $rr\mathbb{N}$. So $rr\mathbb{N}$ has also the following properties:

- has infinitely many objects,
- each map set $rr\mathbb{N}\Big((0,(m,[k_1,\cdots,k_s])),(0,(n,[l_1,\cdots,l_t]))\Big)$ is finite and decidable.

Finally, let's illustrate how to find the $(\mathcal{E}, \mathcal{M})$-factorization of a given map by using the map

$$(1,(1,[2,1,1])) : (0,(1,[2,1,1])) \to (0,(1,[1,1])).$$

Clearly, $(1,(\widehat{1,[2,1},1])) = (0,(4,[0,0]))$ and $(0,(4,[0,0]))$ can be split as

$$(0,(4,[0,0])) = (0,(4,[0,0]))(0,(4,[0,0])).$$

Hence

$$\begin{aligned}(1,(1,[2,1,1])) &= (1,(\widehat{1,[2,1},1]))(1,(1,[2,1,1])) \\ &= (0,(4,[0,0]))(0,(4,[0,0]))(1,(1,[2,1,1])) \\ &= (0,(4,[0,0]))(1,(1,[2,1,1])),\end{aligned}$$

which is the $(\mathcal{E}, \mathcal{M})$-factorization of $(1,(1,[2,1,1]))$ by Theorem 2.3.7. So, $rr\mathbb{N}$ is quite surprising and interesting.

Chapter 6

Conclusions and Further Work

In this thesis, we abstract the notion of partial maps over a system of monics which are part of a factorization system as a range restriction category by axiomatizing both domains and ranges of partial maps and study their properties. Range restriction categories fill out the theory of restriction categories in a particular direction. In this chapter, we provide some concluding remarks and considerations for further work.

6.1 Main Results

The main results we obtained in this thesis are summarized as follows.

1. **Range Restriction Categories are Equivalent to Partial Map Categories**

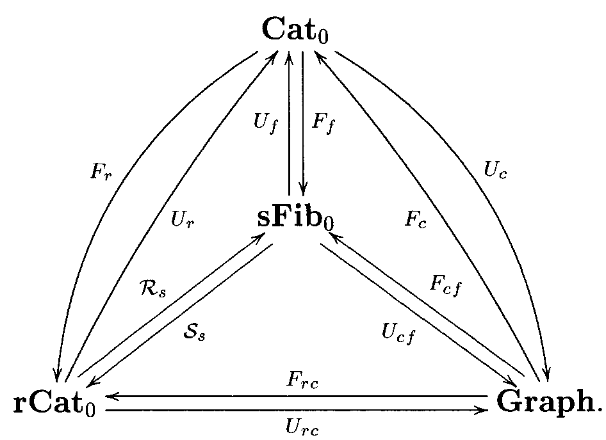
   Theorem 2.2.5 states that if $\mathbf{C}$ is a category with $(\mathcal{E}, \mathcal{M})$-factorization system which is pullback stable along $\mathcal{M}$-maps with $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$ and if $\mathbf{C}$ has pullbacks along $\mathcal{M}$-maps, then $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a range restriction category with the split restriction structure.

   Theorem 2.3.7 states that if $\mathbf{C}$ is a range restriction category, then $\mathsf{Total}(\mathsf{Split}_E(\mathbf{C}))$ admits the $(\mathcal{E}, \mathcal{M})$-factorization system which is pullback stable along $\mathcal{M}$-maps and has pullbacks along $\mathcal{M}$-maps, where $\mathcal{E} = \{f : X \to Y \text{ in } \mathsf{Total}(\mathsf{Split}_E(\mathbf{C})) \mid \widehat{f} = 1_Y\}$ and $\mathcal{M} = \{m : X \to Y \text{ in } \mathsf{Total}(\mathsf{Split}_E(\mathbf{C})) \mid \exists\, r : Y \to X \text{ in } \mathsf{Split}_E(\mathbf{C}),\ rm = 1_X \text{ and } \overline{mr} = mr\}$. Thus, each range restriction category can be embedded into a partial map category.

Theorem 2.3.8 states that the category of range restriction categories with split restriction is 2-equivalent to the category of $\mathcal{M}$-stable factorization systems.

2. **Restriction Categories and Stable Meet Semilattice Fibrations**

Each restriction category gives rise to a stable semilattice fibration (Lemma 3.2.3). Conversely, each stable semilattice fibration produces a restriction structure (Proposition 3.2.4). In this direction, relationships established in this thesis can be summarized by the following commutative diagram of adjunctions:



3. **Range Restriction Categories and Range Stable Meet Semilattice Fibrations**

Each range restriction category gives rise to a range stable meet semilattice fibration (Lemma 4.1.2). Conversely, for each range stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, $\mathcal{S}_{rs}(\mathbf{X})$ is a range restriction category (Proposition 4.1.4). We also showed that there is an adjunction:

$$\mathbf{rrCat}_0 \underset{\mathcal{R}_{rs}}{\overset{\mathcal{S}_{rs}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{rsFib}_0$$

with a faithful functor $\mathcal{R}_{rs}$ (Theorem 4.1.11).

4. **Free Range Restriction Categories over Directed Graphs**

   We provide an explicit construction of the free range restriction categories over directed graphs using deterministic based trees (Theorem 5.3.4) and give an explicit description of the free range restriction category generated by a single endomap.

6.2 Further Work

We list here some possible directions for future work and some questions to which we would like to know the answers.

1. The biggest gap in this thesis is the construction of the free range restriction categories over arbitrary categories. This construction would provide not only a left adjoint to the forgetful functor $U_{rr} : \mathbf{rrCat}_0 \to \mathbf{Cat}_0$, but also a class of examples of range restriction categories. For general reasons, the adjoint exists. However, giving a detailed description was beyond the scope of the thesis.

2. A crucial technique of the thesis was the use of fibrations to construct (range) restriction categories. The structure of these categories of fibrations was not studied in any detail.

3. The thesis describes a number of adjunctions (e.g., Theorems 3.2.28 and 5.3.4),

but does not study the monadicity or Eilenberg-Moore algebras of those adjunctions. This is a gap that could be filled in the future.

4. Each category $\mathbf{C}$ gives rise to a topological functor $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ (Proposition 3.2.19). This functor gives the free stable meet semilattice fibration $\Delta_{\mathbf{C}}$ which produces the free restriction category over $\mathbf{C}$. The connection between restriction categories and topology needs further investigation.

5. Let $\mathbf{X}$ be a restriction category, and A an object of $\mathbf{X}$. $\varepsilon_A : RA \to A$ is a *classifier* at A if ε_A is a restriction retraction and every map with codomain A factorizes through ε_A by a unique total map. A restriction category is a *classified restriction category* if it has a classifier at every object. In [8], Cockett and Lack studied classified restriction categories and showed that the category of classified restriction categories in which the restriction idempotents split is equivalent to the category of classified $\mathcal{M}$-categories (i.e., having an $\mathcal{M}$-partial map classifier).

   We call a range restriction category *classified* if it is classified as a restriction category. It would be interesting to explore the analogous results proven in [8] for range restriction categories.

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