Characterizing Spaces by Disconnection Properties

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By

Chang-Cheng Yang

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SUMMARY OF DISSERTATION

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of the requirements for the

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Characterizing Spaces by Disconnection Properties

In curve theory there is a long history of taking some interesting disconnection property and then studying the class of spaces determined by this property. In this thesis we study the spaces in which every countably infinite set disconnects.

The *disconnection number*, $D^s(X)$, of a connected space X is defined to be the smallest cardinal number κ such that X becomes disconnected upon removal of any set A with $|A| = \kappa$ and $|X \setminus A| \geq 2$ provided such κ exists. We write $X \in D_{\aleph_0}$ if $D^s(X) \leq \aleph_0$ and call X a $D_{\aleph_0}$-*space*. We write $X \in D_{s\omega}$ if $X \in D_{\aleph_0}$ and if each separator F of X between any two points a and b of X contains a separator between a and b consisting of finitely many points and call X a $D_{s\omega}$-*space*.

Stone [St] obtained a characterization of connected, locally connected, separable, metric $D_{\aleph_0}$-spaces. It is a corollary of Stone's theorem that every locally connected, separable, metric $D_{\aleph_0}$-space X is a D_n-space for some integer n. Stone asked for an independent proof of this fact (*i.e.*, one which does not rely on Stone's characterization theorem). We present a characterization theorem of these spaces and in the process we obtain an answer to Stone's question.

We obtain a structure theorem for the class of connected, Hausdorff spaces in $D_{s\omega}$: If X is a connected, Hausdorff space in $D_{s\omega}$, then there exists a weaker topology for X which makes X a locally connected, Tychonoff, $D_{s\omega}$-space. Under this weaker topology X is the union of a rim-finite generalized R-tree and a finite set. If X is a connected, semi-colocally connected, separable metric $D_{s\omega}$-space, then X is hereditarily locally connected and, hence, X is the union of a R-tree and a finite set. If X is a non-degenerate, countably compact, connected, separable, Hausdorff, $D_{s\omega}$-space, then there exists a weaker topology for X which makes X a metric graph.

For the class of non-metric continua in $D_{\aleph_0}$ we give a characterization theorem as follows: A Hausdorff continuum X is a $D_{\aleph_0}$-space if and only if X is a generalized graph. This generalizes a theorem of Nadler in the metric case.

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Abstract

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Stone [St] obtained a characterization of connected, locally connected, separable, metric $D_{\aleph_0}$-spaces. It is a corollary of Stone's theorem that every locally connected, separable, metric $D_{\aleph_0}$-space X is a D_n-space for some integer n. Stone asked for an independent proof of this fact (*i.e.*, one which does not rely on Stone's characterization theorem). We present a characterization theorem of these spaces and in the process we obtain an answer to Stone's question.

We obtain a structure theorem for the class of connected, Hausdorff spaces in $D_{s\omega}$: If X is a connected, Hausdorff space in $D_{s\omega}$, then there exists a weaker topology for X which makes X a locally connected, Tychonoff, $D_{s\omega}$-space. Under this weaker topology X is the union of a rim-finite generalized R-tree and a finite set. If X is a connected, semi-colocally connected, separable metric $D_{s\omega}$-space, then X is hereditarily locally connected and, hence, X is the union of a R-tree and a finite set. If X is a non-degenerate, countably compact, connected, separable, Hausdorff, $D_{s\omega}$-space, then there exists a weaker topology for X which makes X a metric graph.

For the class of non-metric continua in $D_{\aleph_0}$ we give a characterization theorem as follows: A Hausdorff continuum X is a $D_{\aleph_0}$-space if and only if X is a generalized graph. This generalizes a theorem of Nadler in the metric case.

The connectivity degree of a space is introduced and its relation with disconnection number is discussed.

This thesis is dedicated to the memory of my father

YANG Zehua (1931 - 1994)

For his guidance and sacrifices in my life

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Contents

Introduction

In topology a basic problem is to determine when two spaces are homeomorphic. Topologists have developed many tools to do this. In dimension theory one assumes a space can be separated between each closed set and each point outside that set by a subset of certain integral degree of complexity, called its dimension. One gets the class of one dimensional continua when these separators are homeomorphic to subsets of the Cantor set. Curve theory attempts to stratify one dimensional continua which admit such separators which are also in some sense small. Whyburn [Wh1] developed the beautiful and useful cyclic element theory which considers the structure of locally connected continua determined by their single point separators. This theory had been extended considerably by Whyburn [Wh2], Cornette [Cor], Lehman [Leh], Tymchatyn, Nikiel, Tuncali [NTT2], and many others. A tree can be characterized as a locally connected continuum in which every two distinct elements are separated by a third element. A rim-finite (resp. rim-countable) continuum is one in which we can choose separators to be finite (resp. countable, see for example [Wh1] or [Ku]). There is even a well-developed theory of spaces of rim-type $\leq \alpha$ for a countable ordinal α which is analogous to that of one dimensional spaces. There exist, for example, universal objects (non-compact) which are analogues of the Menger curve [M-T]. In some classes of spaces all separators contain "nice" separators. For example, every separator of a locally connected, metric space between two points contains a closed irreducible separator between those points (Mazurkiewicz's Theorem) and every separator of a hereditarily locally connected continuum even contains a metrizable separator [NTT1].

Dimension theory was not put on a firm footing until the 1920's although Poincaré in 1912 had deeply perceived the inductive nature of dimension and the possibility of disconnecting a space by certain subsets. Poincaré was not alone. Janiszewski in 1912 characterized simple arcs as metric continua with exactly two non-separating points. Later, A. J. Ward in 1936 characterized the real line topologically as a connected, locally connected, separable metric space which is separated by each of its points into exactly two components. Bing in 1946 characterized the 2-sphere as a locally connected metric continuum in which no pair of points separates it, but every simple closed curve does separate it.

More generally in curve theory one often decides on an interesting disconnection property and investigates the class of spaces which it characterizes.

Nadler [Na1] defined the *disconnection number*, $D^s(X)$, of a connected space X to be the smallest cardinal number κ such that X becomes disconnected upon removal of any set A with $|A| = \kappa$ and $|X \setminus A| \geq 2$ provided such κ exists. We write $X \in D_\kappa$ if $D^s(X) \leq \kappa$ and call X a *D_κ-space*. We write $X \in D_{s\kappa}$ if $X \in D_\kappa$ and if each separator F of X between any two points a and b contains a separator of X between a and b consisting of at most κ points.

Almost forty years ago, M. Shimrat [Sh, Theorem 2] characterized locally connected, connected, separable, metric D_1-spaces as locally connected, connected, separable, metric spaces which have no endpoints, contain no simple closed curves and are locally arc connected. Applying Shimrat's result, A. H. Stone [St] gave a characterization of the class of locally connected, connected, separable, metric space in $D_{\aleph_0}$ as follows: Every locally connected, connected, separable, metric $D_{\aleph_0}$-space X is a D_n-space for some finite integer n, and consists of a connected finite linear graph L, together with a countable family of pairwise disjoint open ramifications (i.e., locally connected D_1-spaces) such that these ramifications are open subsets of $X \setminus L$, and the frontier of each in X is a single point of L. In [Na1] Nadler proved that every metric $D_{\aleph_0}$-continuum is a D_n-space for some finite n, and, hence, that X is a graph. In [Pi], Pierce gave an example of a subspace X of $\Re^3$ with $\dim(X) = 1$ and $D^s(X) = \aleph_0$. Pierce's example is necessarily not locally connected and not locally compact. In [Gl], Gladdines gave an example of a metric hereditarily locally connected space X with $\dim(X) = 1$ and $D^s(X) = \aleph_0$. Gladdines' example is necessarily not separable.

In this thesis we shall study certain classes of $D_{\aleph_0}$-spaces motivated by Pierce's and Gladdines' examples. In particular, we give another proof of Stone's theorem, we study the structure of $D_{s\omega}$-spaces and extend Nadler's theorem to the non-metric case. In all of this local connectedness plays a central role. The layout of this thesis is as follows.

In Chapter 1 we present some necessary definitions and related theorems which will be used in the following chapters.

In Chapter 2 we investigate locally connected, connected, separable, metric spaces which have disconnection numbers less than or equal to $\aleph_0$. We show that locally connected, connected, separable, metric spaces X with $D^s(X) \leq \aleph_0$ are rim-countable, hereditarily locally connected, σ-compact ANRs which contain only finitely many simple closed curves and finitely many endpoints and, hence, X becomes a R-tree upon removal of finitely many selected points. Conversely, if X is a locally connected, connected, separable, metric space which contains only finitely many simple closed curves and is the union of a R-tree Y with finitely many endpoints and a finite set Z, then X is in $D_{\aleph_0}$. Stone [St] had obtained a characterization of these spaces. As a corollary he obtained that each such $D_{\aleph_0}$-space is D_n for some positive integer n. He asked for an independent proof of this corollary which our work provides. The work in this chapter can be regarded as a special case of the topics in Chapter 3. We have chosen to keep it separate because it is a relatively simple setting for the ideas of Chapter 3.

In Chapter 3 we introduce $D_{s\omega}$-spaces and study their structure. We say a space X is a $D_{s\omega}$-space if $X \in D_{\aleph_0}$ and if each separator F of X between any two points a and b contains a finite separator of X between a and b. We have the following structure theorem: If X is a connected, Hausdorff space in $D_{s\omega}$, then there exists a weaker topology for X which makes X a locally connected, Tychonoff, $D_{s\omega}$-space. Under this weaker topology X is the union of a rim-finite generalized R-tree and a finite set. If X is a connected, semi-colocally connected, separable metric $D_{s\omega}$-space, then X is hereditarily locally connected and, hence, X is the union of a R-tree and a finite set by the work in Chapter 2. If X is a non-degenerate, countably compact, connected, separable, Hausdorff, $D_{s\omega}$-space, then there exists a weaker topology for X which makes X a metric graph.

Nadler [Na1] had proved that a connected, compact, metric $D_{\aleph_0}$-space is a graph. In Chapter 4 we extend Nadler's result to the non-metric case: A Hausdorff continuum X is a generalized graph if and only if $D^s(X) \leq \aleph_0$.

In Chapter 5 we introduce the connectivity degree of a space and study its relation with disconnection number. The connectivity degree of a space is the maximal number of independent connections between some two points of the space. We use Tymchatyn's n-open connections theorem, which generalizes Whyburn's n-arc theorem, to show that if X is a locally connected and connected separable metric space with $D^s(X) \leq \aleph_0$ then X has finite connectivity degree.

In Chapter 6 we give some examples around the theory we have established in the previous chapters. In particular, we show that for any $n \in \{1, 2...., \infty\}$ there is a connected separable metric space Z with $D^s(Z) = 1$ and $\dim(Z) = n$ (Example 6.1). By the results of Chapter 3 this space is homeomorphic to the real line in a coarser topology. Hence, in general $D_{\aleph_0}$ has little to do with dimension. Example 6.12 shows that the n-open connections theorem fails for non-locally connected spaces and this example also gives a negative answer to a question in [Tym].

Chapter 1

Preliminaries

In this chapter we state some definitions and related theorems which will be used in the following chapters. A *topological space* is a pair of $(X, \mathcal{T})$ consisting of a set X and a collection $\mathcal{T}$ of subsets of X satisfying the following conditions: (T1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$. (T2) If $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$. (T3) If $\mathcal{A} \subset \mathcal{T}$, then $\bigcup \mathcal{A} \in \mathcal{T}$. The set X is called a *space*, the elements of X are called *points* of the space, each element $U \in \mathcal{T}$ is called an *open set* of X and its complement $X \setminus U$ is called a *closed set* of X. The collection $\mathcal{T}$ is called a *topology* on X. Let A be a subset of a topological space X. The *closure* of A, denoted by $cl(A)$ (or $cl_X(A)$), is the smallest closed set containing A. The *interior* of A, denoted by A^0 (or $int(A)$), is the largest open set contained in A. We define the *boundary* of A to be the set $bd(A) = cl(A) \cap cl(X \setminus A)$. We denote the cardinality and the complement of A by $|A|$ and $A^c = X \setminus A$ respectively. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces. A mapping f of X to Y is called *continuous* if $f^{-1}(U) \in \mathcal{T}$ for any $U \in \mathcal{T}'$. Throughout this thesis all mappings are continuous.

1.1 Separating Points

In this section, unless stated otherwise, X denotes a non-degenerate, connected, T_1 space.

Let A, B and S be subsets of a topological space X. If $X \setminus S = P \cup Q$ where $A \subset P$, $B \subset Q$ and $cl(P) \cap Q = P \cap cl(Q) = \emptyset$, we then say that S *separates A and B in X*. A set which separates two nonempty subsets of X is called a *separator* of X. If $p \in X$ and if $\{p\}$ is a separator of X between some two points in the component of p in X, then p is called a *separating point* of X. A point p of a topological space X is called *a local separating point* of X provided there exists an open neighborhood U of p such that $\{p\}$ separates U between some two points of the component of U containing p. We say in this case that p is *a local separating point of X with respect to U.*

Lemma 1.1.1 *Let p be a local separating point of X with respect to an open set U in X. Then $V \setminus \{p\}$ is disconnected for every open set V such that $p \in V \subset U$.*

Proof. We have a separation $U \setminus \{p\} = P \cup Q$ where P and Q each contain some points of the component of U containing p. Let V be open such that $p \in V \subset U$. Suppose $V \setminus \{p\}$ is connected. Then $V \setminus \{p\}$ is either in P or in Q. Assume $V \setminus \{p\} \subset P$. Hence $V \cap Q = \emptyset$. It follows that $p \notin cl(Q)$, i.e., Q is open and closed in U. This contradicts that Q contains some points of the component of U containing p. Therefore $V \setminus \{p\}$ is disconnected.

Lemma 1.1.2 *If G is any uncountable set of separating points of a separable, connected, T_1 space X then some two points of G are separated in X by a third point of G.*

Proof. Let $G = \{p_\gamma\}_{\gamma \in \Gamma}$ where $|\Gamma|$ is uncountable and $p_\gamma = p_\beta$ iff $\gamma = \beta$. Suppose that for each $\gamma \in \Gamma$ we have a separation $X \setminus \{p_\gamma\} = U_\gamma \cup V_\gamma$ with $G \setminus \{p_\gamma\} \subset U_\gamma$. Then for each pair α, $\beta \in \Gamma$, $\alpha \neq \beta$, $X = (U_\alpha \cup U_\beta) \bigcup (V_\alpha \cap V_\beta)$ is a separation of X unless $V_\alpha \cap V_\beta = \emptyset$. Since X is connected $V_\alpha \cap V_\beta = \emptyset$ for $\alpha \neq \beta$. Hence, X contains uncountably many mutually disjoint open sets $\{V_\gamma\}_{\gamma \in \Gamma}$ which contradicts that X is a separable space. Therefore, there exists $\gamma_0 \in \Gamma$ such that $\{p_{\gamma_0}\}$ separates some two points of G in X.

Theorem 1.1.3 *If X is a connected T_1 space and $p \in X$ then the following statements are equivalent:*

(a) p is a separating point of X.

(b) $X \setminus \{p\} = U \cup V$ where U and V are disjoint open sets, $cl(U) = U \cup \{p\}$, $cl(V) = V \cup \{p\}$ and $cl(U)$ and $cl(V)$ are connected.

(c) $X = M \cup N$ where M and N are non-degenerate closed and connected sets such that $M \cap N = \{p\}$.

Proof. *(a) implies (b).* Let p be a separating point of X. Then $X \setminus \{p\} = U \cup V$ where $cl(U) \cap V = U \cap cl(V) = \emptyset$ and U and V are nonempty. Since $X \setminus \{p\}$ is open, so are U and V. Next, $U \cup \{p\} = X \setminus V$ is closed, so $cl(U) \subset U \cup \{p\}$. If $p \notin cl(U)$ then $cl(U) \subset X \setminus (\{p\} \cup V) = U$. Hence, $cl(U) = U$ which is a closed and open proper subset in X, contrary to the connectivity of X. So $p \in cl(U)$ and, hence, $cl(U) = U \cup \{p\}$. Finally suppose $cl(U) = A \cup B$ where A and B are disjoint closed subsets of X such that $p \in A$. Since $B \bigcap cl(A \cup V) = B \bigcap (A \cup V \cup \{p\}) = \emptyset$, $X = B \cup (A \cup V)$ will be a separation of X unless $B = \emptyset$. Therefore, $cl(U)$ is connected. Similarly, $cl(V)$ is connected.

(b) implies (c). Let $M = U \cup \{p\}$ and $N = V \cup \{p\}$ as in (b). Then M and N are non-degenerate closed and connected sets such that $M \cap N = \{p\}$ as required.

(c) implies (a). Let $X = M \cup N$ be given as in (c). Put $A = M \setminus \{p\}$ and $B = N \setminus \{p\}$. Then $X \setminus \{p\} = M \cup N \setminus \{p\} = (M \setminus \{p\}) \cup (N \setminus \{p\}) = A \cup B$, $cl(A) \cap B \subset M \cap (N \setminus \{p\}) = \emptyset$ and $A \cap cl(B) \subset (M \setminus \{p\}) \cap N = \emptyset$. Therefore $X \setminus \{p\} = A \cup B$ is a separation and, hence, p is a separating point of X.

Let P be a set. A *partial ordering* of P is a relation $\prec$ on P such that: (a) if $x \prec y$ and $y \prec z$ then $x \prec z$; (b) $x \prec y$ and $y \prec x$, if and only if $x = y$. A pair $(P, \prec)$ where P is a set and $\prec$ is a partial ordering of P is called a *partially ordered set.* An ordering $\prec$ is said to be *linear* if the following supplementary condition is satisfied: (c) for every x, $y \in X$, either $x \prec y$ or $y \prec x$. A subset of P on which $\prec$ is a linear ordering is called a *chain* in the ordered set $(P, \prec)$.

Hausdorff Maximality Principle ([Ward], p.8) *If X is a partially ordered set then every chain in X is contained in a maximal chain in X.*

A compact, connected, Hausdorff space is called a *continuum.*

Theorem 1.1.4 (Non-Separating Point Existence Theorem) *A non-degenerate continuum has at least two non-separating points.*

Proof. Suppose X is a continuum with at most one non-separating point. Let $p \in X$ be the non-separating point of X if one exists or an arbitrary point of X, otherwise. Then, each $x \in X \setminus \{p\}$ is a separating point of X. By Theorem 1.1.3 let $X = M_x \cup N_x$ where M_x and N_x are non-degenerate subcontinua such that $p \in M_x$ and $M_x \cap N_x = \{x\}$.

Claim *For every two distinct points x, $y \in X \setminus \{p\}$, if $x \in N_y$ then $N_x \subset N_y \setminus \{y\}$.*

Proof of Claim. If $x \in N_y$ then $x \notin M_y$. So $M_y \subset (M_x \cup N_x) \setminus \{x\}$. The sets $M_x \setminus \{x\}$ and $N_x \setminus \{x\}$ are disjoint and $p \in M_y \cap (M_x \setminus \{x\})$. Then $M_y \subset M_x \setminus \{x\}$ since M_y is connected. So $N_x = (X \setminus M_x) \cup \{x\} \subset X \setminus M_y$. It follows that $N_x \subset N_y \setminus \{y\}$ as claimed.

Let $\mathcal{N} = \{N_x\}_{x \in X \setminus \{p\}}$ be partially ordered by inclusion, i.e., sets $N_x \leq N_y$ iff $N_x \subset N_y$. Applying the Hausdorff Maximality Principle, there exists a maximal chain $\mathcal{N}_0 \subset \mathcal{N}$. We index $\mathcal{N}_0 = \{N_\alpha\}_{\alpha \in A}$. Since $\mathcal{N}_0$ is a chain it has the finite intersection property. Since X is compact, $\cap \mathcal{N}_0 = \bigcap_{\alpha \in A} N_\alpha \neq \emptyset$. Pick a point $q \in \cap \mathcal{N}_0$. Then $N_q \subset N_\alpha$ for all $\alpha \in A$ by the Claim. By the maximality of $\mathcal{N}_0$, $N_q \in \mathcal{N}_0$ and N_q is the smallest element of $\mathcal{N}_0$.

Let $x \in N_q \setminus \{q\}$. By the claim we have $N_x < N_q$ and, hence, $N_x \leq N_\alpha$ for all $\alpha \in A$. By the maximality of $\mathcal{N}_0$, $N_x \in \mathcal{N}_0$. But $N_q \leq N_x$ which is a contradiction. The theorem is proved.

Corollary 1.1.5 *If X is a continuum then no proper connected subset of X contains all of the non-separating points of X.*

Proof. Suppose there exists a proper connected subset Y of X which contains all of the non-separating points of X. Let $x \in X \setminus Y$. Then we have a separation $X \setminus \{x\} = U \cup V$. Since Y is connected we may assume $Y \subset U$. Then V does not contain any non-separating point of X. But $cl(V) = V \cup \{x\}$ is a subcontinuum. Applying Theorem A.4 we pick a point $p \in cl(V) \setminus \{x\} = V$ which is a non-separating point of $cl(V)$, i.e., $cl(V) \setminus \{p\}$ is connected. Since $cl(U) \cap (cl(V) \setminus \{p\}) = \{x\}$, $X \setminus \{p\} = cl(U) \cup (cl(V) \setminus \{p\})$ is connected and, hence, V contains a non-separating point p of X. This is a contradiction. Therefore, no proper connected subset of X contains all of the non-separating points of X.

Let X be a connected, Hausdorff space and let a and b be two points of X. Let $E_X(a,b) = \{x \in X : x \textit{ separates } a \textit{ and } b \textit{ in } X\} \cup \{a, b\}$ and we define a *natural order* on $E_X(a,b)$ as follows: For each $x \in E_X(a,b) \setminus \{a, b\}$ let $X = L_x \cup M_x$ where L_x and M_x are proper subcontinua of X such that $L_x \cap M_x = \{x\}$ and $a \in L_x$ and $b \in M_x$. Let $L'_x = L_x \cap E_X(a,b)$ and $M'_x = M_x \cap E_X(a,b)$. For $x, y \in E_X(a,b) \setminus \{a, b\}$ we define

$$(*) \qquad x \leq y \iff y \in M_x \text{ and}$$
$$a \leq z \leq b \text{ for every } z \in E_X(a,b)$$

Theorem 1.1.6 *Let X be a connected Hausdorff space and a and b two points of X. The relation $\leq$ is a linear ordering on $E_X(a,b)$ and the order topology on $E_X(a,b)$ is* *coarser than the subspace topology on $E_X(a,\ b)$ inherited from X.*

Proof. Claim 1 *For each $x \in E_X(a,b) \setminus \{a,\ b\}$ $L'_x = \{y \in E_X(a,b):\ y \leq x\}$ and $M'_x = \{y \in E_X(a,b):\ x \leq y\}$.*

Proof of Claim 1. For $x,\ y \in E_X(a,b)$, since $y < x$ implies $x \in M_y$ or $x \notin L_y$. This implies $L_y \subset (L_x \cup M_x) \setminus \{x\}$ and, hence, implies $L_y \subset L_x \setminus \{x\}$. So $y \in L_x$ or $y \in L'_x$. Next suppose $y \in L'_x$ ($y \neq x$). This implies $y \notin M_x$ and, hence, implies $M_x \subset (L_y \cup M_y) \setminus \{y\}$ which implies $M_x \subset M_y \setminus \{y\}$ or $x \in M_y$. So $y \leq x$. Therefore, $L'_x = \{y \in E_X(a,b):\ y \leq x\}$. The second statement is clear by definition of (*).

Claim 2 *the relation $\leq$ is a linear ordering on $E_X(a,b) \setminus \{a,\ b\}$.*

Proof of Claim 2. (i) $x \leq x$ since $x \in M_x$. (ii) If $x \leq y$ and $y \leq x$. By Claim 1 $y \in L_x \cap M_x$. Then $y = x$. (iii) If $x \leq y$ and $y \leq z$. Suppose $z \neq y$. By Claim 1 $M_z \subset M_y \setminus \{y\}$ and $M_y \subset M_x \setminus \{x\}$. Thus $z \in M_x$ or $x \leq z$. (iv) For any pair $x,\ y \in X$ we have either $y \in L_x$ or $y \in M_x$. That is, by Claim 1, either $y \leq x$ or $x \leq y$. Therefore, $\leq$ is a linear order on $E_X(a,b)$.

Since $a \leq z \leq b$ for every $z \in E_X(a,b)$, a and b are the smallest element and largest element of $E_X(a,b)$ respectively. Hence, by Claim 2, the relation $\leq$ is a linear ordering on $E_X(a,b)$.

Finally suppose $\mathcal{T}$ is the subspace topology on $E_X(a,b)$ inherited from X. The elements of a subbase for the order topology $\mathcal{O}$ of $E_X(a,b)$ each have one of the following forms:

$$[a,\ x) = L_x \setminus \{x\} \text{ and } \quad (x,\ b] = M_x \setminus \{x\}.$$

All are elements of $\mathcal{T}$ and, hence, the identity function

$id:\ (E_X(a,b),\ \mathcal{T}) \longrightarrow (E_X(a,b),\ \mathcal{O})$ is continuous. This completes the proof of Theorem 1.1.6.

A subset S of a space X is called *an irreducible separator of X between two subsets A and B* provided S separates A and B in X and there exists no proper subset of S which separates X between A and B. We say a space X is *hereditarily normal* if every subspace of X is normal.

Lemma 1.1.7 *Every separator of a hereditarily normal space X between two subsets A and B of X contains a closed separator of X between A and B.*

Proof. Let S be a separator of X between two subsets A and B. Let $X \setminus S = P \cup Q$ where P and Q are separated sets, $A \subset P$ and $B \subset Q$. Since X is hereditarily normal, there exist two disjoint open subsets U and V of X containing P and Q respectively. Then $S_0 = X \setminus (U \cup V) \subset S$ is a closed separator of X between A and B.

Lemma 1.1.8 (Mazurkiewicz's Theorem) *Let X be a locally connected, hereditarily normal space. If $F \subset X$ separates two points a and b in X, then F contains an irreducible closed subset F_0 which separates a and b in X.*

Proof. By Lemma 1.1.7 we may assume F is closed. Let C be the component of $X \setminus F$ containing a. Since X is locally connected, C is open. Now $Bd(C) = cl(C) \setminus C \subset F$ and $b \in X \setminus cl(C)$. Let D be the component of $X \setminus cl(C)$ containing b. Then D is open and $Bd(D) = cl(D) \setminus D \subset cl(C) \setminus C \subset F$. Put $F_0 = Bd(D)$. Then $X \setminus F_0 = D \cup (X \setminus cl(D))$ is a separation and $a \in C \subset X \setminus cl(D)$ and $b \in D$. If $x \in F_0$ then $x \in Bd(C) \cap Bd(D)$ and $C \cup \{x\} \cup D$ is a connected subset of $(X \setminus F_0) \cup \{x\}$ containing a and b. Therefore, F_0 is the required set.

Let Λ be a set and $\leq$ a relation on X. We say that the relation $\leq$ *directs* X if $\leq$ is reflexive, transitive and for any $\lambda_1, \lambda_2 \in \Lambda$ there exists a $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. A *net in a topological space* X is an arbitrary function from a nonempty directed set to the space X. Nets will be denoted by $\{x_\lambda\}_{\lambda \in \Lambda}$ where x_λ is the point of X assigned to the element λ of the directed set Λ. We say a net $\{x_\lambda\}_{\lambda \in \Lambda}$ is *frequently in every neighborhood of a point* x of a space X if for every neighborhood U of x and for every λ there exists a $\lambda' \geq \lambda$ such that $x_{\lambda'} \in U$. We say a net $\{x_\lambda\}_{\lambda \in \Lambda}$ is *eventually in every neighborhood of a point* x of a space X if for every neighborhood U of x there exists a λ_0 such that $x_\lambda \in U$ for each $\lambda \geq \lambda_0$.

Theorem 1.1.9 *Let X be a connected, locally connected, T_1, regular space and let a and b be two points of X. Then $E_X(a,b)$ is compact and the order topology on $E_X(a, b)$ introduced by $\leq$ and the subspace topology on $E_X(a, b)$ are identical.*

Proof. Let $\{y_\alpha\}_{\alpha \in A}$ be a net in $E_X(a,b)$. Suppose there exists no cluster point for this net. Then for each $x \in X$ there exists a connected neighborhood U_x of x and $\alpha(x) \in A$ with $y_\alpha \notin U_x$ for each $\alpha \geq \alpha(x)$. Since X is connected there exists a finite chain, say $U_{x_1}, \cdots, U_{x_n}$

from a to b. Let $U = \bigcup_{i=1}^{n} U_{x_i}$. Let $\alpha_0 \in A$ with $\alpha_0 \geq \alpha(x_i)$ for each $i \in \{1, \cdots, n\}$. If $\alpha \in A$ with $\alpha \geq \alpha_0$ then $y_\alpha \notin U$, i.e., y_α does not separate X between a and b which is a contradiction. So every net in $E_X(a,b)$ has a cluster point y. Next we show that y is in $E_X(a,b)$. Suppose $y \notin E_X(a,\ b)$ and let $C \subset X \setminus \{y\}$ be the component containing a and b. Since X is locally connected C is open. As above we can find a finite chain $\mathcal{C}$ of connected open sets from a to b with $cl(\cup\mathcal{C}) \subset C$. Then $y \notin cl(\cup\mathcal{C})$ and $E_X(a,b) \subset cl(\cup\mathcal{C})$ since $\cup\mathcal{C}$ is connected. It follows that $y \notin cl(E_X(a,b))$ which is a contradiction. Therefore, $E_X(a,b)$ is compact.

Suppose $\mathcal{T}$ is the subspace topology on $E_X(a,b)$ and $\mathcal{O}$ the order topology on $E_X(a,b)$ introduced by $\leq$. By Theorem 1.1.6 the identity function

$id: \ (E_X(a,b),\ \mathcal{T}) \longrightarrow (E_X(a,b),\ \mathcal{O})$ is continuous. Since $E_X(a,b)$ is compact in $\mathcal{T}$, the identity on $E_X(a,b)$ is a homeomorphism onto $(E_X(a,b),\ \mathcal{O})$. This completes the proof of Theorem 1.1.9.

A subset G of X is said to be *saturated* provided that if $g \in G$ and p is any point of $X \setminus \{g\}$ there exists at least one point q in G which separates p and g in X. A point p is said to have *potential order less than or equal to n in X, for some nonnegative integer n, relative to G* provided there exists a neighborhood basis $\{U_\alpha\}$ of open subsets in X at $\{p\}$ such that for each α, $bd(U_\alpha)$ is a subset of at most n points of G. If p is of potential order less than or equal to n in X relative to G but not of potential order less than or equal to n-1 in X relative to G, p is said to be of *potential order n in X relative to G.*

The following theorem is due to Whyburn [Wh1, Theorem 2.2, p.45].

Theorem 1.1.10 *Each set G of separating points of a separable metric space X contains a saturated subset Q such that $G \setminus Q$ is countable and each point of Q is of potential order 2 in X relative to Q and separates X into exactly two components.*

1.2 Dimension and Rim-Countable Spaces

In this section, unless stated otherwise, let X denote a non-degenerate, separable, metric space.

Definition of dimension n. *The empty set and only the empty set has dimension -1. A space X has dimension $\leq n$ ($n \geq 0$) at a point p if p has a basis of neighborhoods whose boundaries have dimension $\leq n-1$. The space X has dimension $\leq n$ iff X has dimension $\leq n$ at each of its points. We say a space X has dimension n if dim $X \leq n$ is true and dim $X \leq n-1$ is false. Finally, X has dimension ∞ if dim $X \leq n$ is false for each integer n.*

The following three results will be used later. The reader may find the proofs of these results in any book on dimension theory (see for example [H-W]).

Theorem 1.2.1 (The Sum Theorem for 0-dimensional Sets). *A space which is the countable union of 0-dimensional closed subsets is itself 0-dimensional.*

Corollary 1.2.2 *The union of two 0-dimensional subsets of a space X at least one of which is closed is 0-dimensional.*

Theorem 1.2.3 *A subspace C of a space X has dimension $\leq n$ if and only if every point of C has arbitrarily small neighborhoods in X whose boundaries have intersections with C of dimension $\leq n-1$.*

We recall that a space X is said to have *order less than or equal to κ at a point p of X*, denoted by ord$(p, X) \leq \kappa$, for some cardinal number κ provided that X has a neighborhood basis at p of open sets $\{U_\alpha\}$ whose boundaries have cardinality $|bd(U_\alpha)| \leq \kappa$. If X is of order less than or equal to κ at p but not of order less than or equal to κ' at p for each $\kappa' < \kappa$ in X, then X is said to be of *order κ at p*. If X has order $\leq \aleph_0$ at p then X is said to be *rim-countable at p*. If X is rim-countable at each of its points, it is said to be *rim-countable*. Similarly, we say a space X to be *rim-finite* provided X has order $< \aleph_0$ at each of its points.

Lemma 1.2.4 *A separable metric space X is rim-countable if and only if it is the union of two subsets one of which is at most 0-dimensional and the other is countable.*

Proof. Let X be rim-countable, and let $\{U_i\}_{i=1}^{\infty}$ be a basis for X such that $|bd(U_i)| \leq \aleph_0$ for each i. Put $D = \bigcup_{i=1}^{\infty} bd(U_i)$. Then D is countable and $\dim(X \setminus D) \leq 0$ since the sets

$\{U_i \setminus D\}_{i=1}^{\infty}$ are closed and open in $X \setminus D$ and form a basis for $X \setminus D$.

Conversely let D be a countable set with $\dim(X \setminus D) \leq 0$. For $p \in X$, $\dim((X \setminus D) \cup \{p\}) = 0$ by Corollary 1.2.2. Applying Theorem 1.2.3 there exists for each $\epsilon > 0$ an open neighborhood G of p with diameter $< \epsilon$ and $bd(G) \cap (X \setminus D) = \emptyset$, i.e., $bd(G) \subset D$. It follows that $|bd(G)| \leq \aleph_0$. Hence, X has order $\leq \aleph_0$ at p. Since p is arbitrary X is rim-countable.

Theorem 1.2.5 *The union of countably many closed rim-countable sets in X is a rim-countable set.*

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$ where each A_i is closed and rim-countable. Set $A_1^* = A_1$, $A_n^* = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. By Lemma 1.2.4 for each n $A_n^* = B_n \cup D_n$ where dim $(B_n) \leq 0$, $|D_n| \leq \aleph_0$ and $B_n \cap D_n = \emptyset$. Hence, $A = \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} D_n$ and $|\bigcup_{n=1}^{\infty} D_n| \leq \aleph_0$. Observe that each A_n^* is open in A_n and, hence, an F_σ set in X. Then $B_n = A_n^* \cap (\bigcup_{i=1}^{\infty} B_i)$ is an F_σ set in $\bigcup_{n=1}^{\infty} B_n$. By Theorem 1.2.1 $\dim(\bigcup_{n=1}^{\infty} B_n) \leq 0$. It follows from Theorem 1.2.4 that A is rim-countable.

1.3 Absolute Neighborhood Retracts

In this section by a space we mean a separable metrizable space. We say that a space X is an ***absolute neighborhood retract*** (abbreviated ANR) if, for every space Y containing X as a closed subspace there exists a neighborhood U of X in Y such that there exists a continuous function $r : U \longrightarrow X$ such that r is restricted to X is the identity id_X (such a function is called a ***retraction***). It is well-known that a space X is an ANR if and only if for each closed subset A of a space Y, every mapping $f : A \longrightarrow X$ has a continuous extension $F : U \longrightarrow X$ defined on some neighborhood U of A in Y (ANE, [vanM, 1.5.2, p.45]). A space is said to be an ***ANR locally at a point*** p if there exists a neighborhood of p which is an ANR.

The following theorems of Hanner can be found in [Bor, p.96-99].

Theorem 1.3.1 *Every open subspace of an ANR is an ANR.*

Theorem 1.3.2 *Let $X = \bigcup_{i=1}^{\infty} G_i$ where each G_i is an ANR and an open subset of X. Then the space X is an ANR.*

Theorem 1.3.3 *A separable metric space is an ANR if and only if it is locally an ANR at each of its points.*

1.4 Hereditarily Locally Connected Spaces and Convergence Continua

A Hausdorff space is said to be *hereditarily locally connected* provided each of its connected subsets is locally connected (see [Tym1]).

Let $\{K_\lambda\}_{\lambda\in\Lambda}$ be a net of subsets of a topological space X. The *topological upper limit Lim sup* K_λ (respectively *lower limit Lim inf* K_λ) of the net $\{K_\lambda\}_{\lambda\in\Lambda}$ is the set of all points $x \in X$ such that the net $\{K_\lambda\}_{\lambda\in\Lambda}$ is frequently (resp. eventually) in every neighborhood of x. Evidently *Lim inf* $K_\lambda \subset$ *Lim sup* K_λ. If *Lim inf* $K_\lambda =$ *Lim sup* K_λ then the net $\{K_\lambda\}_{\lambda\in\Lambda}$ is said to be *convergent* and the set *Lim sup* K_λ is denoted by *Lim* K_λ. A subcontinuum K of a topological space X is called a *convergence continuum in* X provided there exists a net $\{K_\lambda\}_{\lambda\in\Lambda}$ of continua of X such that *Lim* $K_\lambda = K$, $K_{\lambda'} \cap K_\lambda = K_\lambda$ or $K_{\lambda'} \cap K_\lambda = \phi$ for $\lambda', \lambda \in \Lambda$ and $K_\lambda \cap K = \phi$ for each λ.

The following theorem is due to Frolik [Fr, Corollary 4.5] and Simone [Si, Theorem 3].

Theorem 1.4.1 *A Hausdorff continuum X is hereditarily locally connected if and only if it contains no convergence continuum.*

1.5 Inverse Limits

An *inverse sequence* is a sequence of pairs $(X_i, f_i)_{i=1}^{\infty}$ of spaces X_i, called *coordinate spaces,* and continuous functions f_i: $X_{i+1} \longrightarrow X_i$ called *bonding maps.* The *inverse limit* of $(X_i, f_i)_{i=1}^{\infty}$, denoted by $\varprojlim\, (X_i, f_i)$, is defined by

$$\varprojlim\, (X_i, f_i) = \{(x_i)_{i=1}^{\infty} \in \textstyle\prod_{i=1}^{\infty} X_i\colon\ f_i(x_{i+1}) = x_i \text{ for all } i\}.$$

Let π_i: $\varprojlim\, (X_i, f_i) \longrightarrow X_i$ denote the ith projection map and let $f_{ij} = f_i \circ \cdots \circ f_{j-1} : X_j \longrightarrow X_i$ if $j \geq i+1$.

Lemma 1.5.1 *Let $X = \varprojlim (X_i, f_i)$ then the collection*

$$\{\pi_i^{-1}(U) : U \text{ is open in } X_i \text{ and } i = 1, 2, \cdots\}$$

forms a basis for the topology of X.

Proof. Let U be an open subset in X and let $x = (x_i)_{i=1}^{\infty} \in U$. Since X has the subspace topology inherited from $\prod_{i=1}^{\infty} X_i$ there exist $U_1, \cdots, U_k$ open in $X_{i_1}, \cdots, X_{i_k}$ respectively such that $x \in \bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_j) \subset U$. Let n be a positive integer such that $i_j \leq n$ for each $j \leq k$. All the sets $f_{i_j n}^{-1}(U_j)$ and their intersection $U_n = \bigcap_{j=1}^{k} f_{i_j n}^{-1}(U_j)$ are open in X_n; further, as $f_{i_j n}(x_n) = x_{i_j}$ we have $x_n \in U_n$. Since $\pi_n^{-1} f_{i_j n}^{-1}(U_j) = \pi_{i_j}^{-1}(U_j)$ we obtain $x \in \pi_n^{-1}(U_n) = \pi_n^{-1}(\bigcap_{j=1}^{k} f_{i_j n}^{-1}(U_j)) = \bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_j) \subset U$ which completes the proof of Lemma 1.5.1.

Lemma 1.5.2 *Let $X = \varprojlim (X_i, f_i)$. Then for any subset A of X we have*

$$cl(A) = \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})}) = [\textstyle\prod_{i=1}^{\infty} cl(A_i)] \cap X$$

where $A_i = \pi_i(A)$ for each i.

Proof. Since $f_i \circ \pi_{i+1} = \pi_i$ for each i it follows that $f_i(cl(A_{i+1})) = f_i(cl(\pi_{i+1}(A))) \subset cl(f_i \circ \pi_{i+1}(A)) = cl(\pi_i(A)) = cl(A_i)$ and, hence, $(cl(A_i), f_i|_{cl(A_{i+1})})$ is an inverse sequence. It is easy to see that $\varprojlim (cl(A_i), f_i|_{cl(A_{i+1})}) = [\prod_{i=1}^{\infty} cl(A_i)] \cap X$; moreover, it is a closed subspace of X. Indeed, for every $x = (x_i)_{i=1}^{\infty} \in X \setminus \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$ there exists a $x_i \in X_i \setminus cl(A_i)$ for some i by Lemma 1.5.1, so that $\pi_i^{-1}(X_i \setminus cl(A_i))$ is a neighborhood of x disjoint from $\varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$. Clearly $A \subset \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$, we then have $cl(A) \subset \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$. To complete the proof let $x = (x_i)_{i=1}^{\infty} \in \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$. By Lemma 1.5.1 the collection of all sets $\pi_i^{-1}(U)$, where U is a neighborhood of x_i in X_i and $i \in \{1, 2, \cdots\}$, is a local base at x in X. For every member $\pi_i^{-1}(U)$ of that base we have $x_i \in cl(A_i) \cap U$, so that $A_i \cap U \neq \emptyset$ or $A \cap \pi_i^{-1}(U) \neq \emptyset$. This implies that $x \in cl(A)$, proving that $cl(A) = \varprojlim (cl(A_i), f_i|_{cl(A_{i+1})})$.

Recall that a surjective mapping $f : X \longrightarrow Y$ is said to be *quotient* if $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X. A surjective mapping $f : X \longrightarrow Y$ is said to be *hereditarily quotient* if for each $A \subset Y$ the restriction $f|_{f^{-1}(A)} : f^{-1}(A) \longrightarrow A$ is quotient. Note that the mapping $f : X \longrightarrow Y$ is hereditarily quotient if and only if, for each $y \in Y$ and each open subset U of X containing $f^{-1}(y)$, the set $f(U)$ is a neighborhood of y in

Y (see [Eng, p.134]). All surjective open mappings and surjective closed mappings are hereditarily quotient.

Theorem 1.5.3 *Let $X = \varprojlim (X_i, f_i)$ where each X_i is connected. Then X is connected if one of the following conditions is satisfied:*

(a) each X_i is compact;

(b) each f_i is monotone, surjective and hereditarily quotient.

Proof. Suppose the condition (a) holds. For each positive integer n we define

$$P_n = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \ : \ f_i(x_{i+1}) = x_i \text{ for all } i \leq n\}.$$

Then (1) $P_{n+1} \subset P_n$; (2) $\varprojlim (X_i, f_i) = \bigcap_{n=1}^{\infty} P_n$; (3) P_n is homeomorphic to $\prod_{i=n+1}^{\infty} X_i$ for each n and, hence, is compact and connected. Indeed, for each n we define

$h: \ P_n \longrightarrow \prod_{i=n+1}^{\infty} X_i$ by $h((x_i)_{i=1}^{\infty}) = (x_i)_{i=n+1}^{\infty}$ for each $(x_i)_{i=1}^{\infty} \in P_n$.

Then h is a homeomorphism as desired. Applying (1), (2) and (3) we obtain that X is connected since the intersection of a nest of continua is a continuum.

Now suppose that condition (b) holds. Below we follow the idea of Puzio [Pu]. We shall prove a claim first.

Claim *For each i the projection $\pi_i : X \longrightarrow X_i$ is hereditarily quotient.*

Subclaim 1 *For each i the projection $\pi_i : X \longrightarrow X_i$ is a surjection.*

Proof of Subclaim 1. For $x_i \in X_i$ let $x_j = f_{ji}(x_i) \in X_j$ for $j < i$. Inductively, pick $x_{i+1} \in f_i^{-1}(x_i)$, $x_{i+2} \in f_{i+1}^{-1}(x_{i+1}), \cdots$, we then obtain a sequence $x = (x_i)_{i=1}^{\infty} \in X$ such that $\pi_i(x) = x_i$.

Subclaim 2 *For each i the projection $\pi_i : X \longrightarrow X_i$ is quotient.*

Proof of Subclaim 2. Let A be a subset of X_i such that $\pi_i^{-1}(A)$ is open in X. Suppose that A is not open in X, i.e., there exists an $x_i \in A$ such that $x_i \in Bd(A)$. Note that $f_i^{-1}(x_i) \subset \pi_{i+1}\pi_i^{-1}(A)$. If $f_i^{-1}(x_i) \subset int(\pi_{i+1}\pi_i^{-1}(A))$ then, since f_i is quotient, $x_i \in int(f_i(int(\pi_{i+1}\pi_i^{-1}(A)))) \subset A$ which is a contradiction. Hence, there exists an $x_{i+1} \in Bd(\pi_{i+1}\pi_i^{-1}(A)) \cap f_i^{-1}(x_i)$. This process may be continued inductively to obtain a sequence $x = (x_j) \in X$ such that $x_j \in Bd(\pi_j\pi_i^{-1}(A))$ for each $j \geq i$. Since $X \setminus \pi_i^{-1}(A)$ is closed and $x_j \in cl(\pi_j(X \setminus \pi_i^{-1}(A)))$ for every $j \geq i$. By Lemma 1.5.2, $x \in X \setminus \pi_i^{-1}(A)$ which is in contradiction with $x_i \in A$. This proves Subclaim 2.

Proof of Claim. Now we show that π_i is hereditarily quotient. For $Y_i \subset X_i$ we have

$\pi_i^{-1}(Y_i) = \varprojlim\ (Y_j, f_j|_{Y_{j+1}})$ where

$$Y_j = \begin{cases} f_{ji}(Y_i) & \text{for } j \leq i\ ; \\ f_{ij}^{-1}(Y_i) & \text{for } j > i. \end{cases}$$

Since each mapping $f_j|_{Y_{j+1}}$ for $j \geq i$ is hereditarily quotient, from the proof of Subclaim 2, it follows that the mapping $\pi_i|_{\pi_i^{-1}(Y_i)} : \pi_i^{-1}(Y_i) \longrightarrow Y_i$ is quotient. This completes the proof of Claim.

Finally, we show that X is connected. Suppose there exists a separation $X = U_1 \cup U_2$ where U_1 and U_2 are open, nonempty and disjoint. By the above Claim the mapping $\pi_i : X \longrightarrow X_i$ is hereditarily quotient. Suppose that $A_i = \pi_i(U_1) \cap \pi_i(U_2) = \emptyset$ for some i. Then $U_k = \pi_i^{-1}\pi_i(U_k)$ for $k = 1,\ 2$, and $X_i = \pi_i(U_1) \cup \pi_i(U_2)$. Since π_i is quotient, the sets $\pi_i(U_1)$ and $\pi_i(U_2)$ are open, nonempty and disjoint. This is in contradiction with the connectivity of X_i; thus all sets A_i are not empty.

Clearly, $f_i(A_{i+1}) \subset A_i$. We shall show that $f_i(A_{i+1}) = A_i$. Take $x_i \in A_i$. Let $B_k = f_i^{-1}(x_i) \cap \pi_{i+1}(U_k)$ for $k = 1,\ 2$. Then, $f_i^{-1}(x_i) = B_1 \cup B_2$. To see that $B_1 \cap B_2 = f_i^{-1}(x_i) \cap A_{i+1} \neq \emptyset$ suppose the contrary. Then $\pi_{i+1}^{-1}(B_k) = \pi_{i+1}^{-1} f_i^{-1}(x_i) \cap U_k = U_k \cap \pi_i^{-1}(x_i)$ and this set is open in $\pi_i^{-1}(x_i)$. Since the restriction $\pi_{i+1}|_{\pi_i^{-1}(x_i)} : \pi_i^{-1}(x_i) \longrightarrow f_i^{-1}(x_i)$ is quotient, the sets B_k are open in $f_i^{-1}(x_i)$ for $k = 1,\ 2$ which contradicts the assumption that $f_i^{-1}(x_i)$ is connected since f_i is monotone.

The sequence $(A_i,\ f_i|_{A_{i+1}})_{i=1}^{\infty}$ is an inverse sequence of nonempty spaces with surjective bonding mappings. Thus $\varprojlim\ (A_i, f_i|_{A_{i+1}}) \neq \emptyset$ and is contained in $U_1 \cap U_2$ since the sets U_k are closed, which contracts the assumption that $U_1 \cap U_2 = \emptyset$ and, hence, Theorem 1.5.3 is proved.

Theorem 1.5.4 *Let $X = \varprojlim\ (X_i, f_i)$ where each bonding mapping is monotone and one of the following two conditions is satisfied:*

(a) each X_i is compact;

(b) each f_i is hereditarily quotient. Then

(i) for each i the projection $\pi_i : X \longrightarrow X_i$ is a monotone surjection and

(ii) if every X_i is locally connected then X is locally connected.

Proof. (i). Suppose the condition (a) holds. For $x_i \in X_i$ let $A = \pi_i^{-1}(x_i)$. Since A is compact, applying Lemma 1.5.2, we have $A = \varprojlim\ (A_j, f_j|_{A_{j+1}})$ where $A_j = \pi_j(A)$. Note that $\pi_j \circ \pi_i^{-1}(x_i) = f_{ij}^{-1}(x_i)$ for $j > i$, so that each $\pi_j(A)$ is connected for $j > i$ and, hence, $\pi_j(A)$ is connected for $j \geq 1$ since $f_j \circ \pi_{j+1} = \pi_j$ for each $j \geq 1$. By Theorem 1.5.3, $A = \pi_i^{-1}(x_i)$ is connected.

Suppose the condition (b) holds. For $x_i \in X_i$ we have $\pi_i^{-1}(x_i) = \varprojlim (A_j, f_j|_{A_{j+1}})$ where

$$A_j = \begin{cases} f_{ji}(x_i) & \text{for } j \leq i\,; \\ f_{ij}^{-1}(x_i) & \text{for } j > i. \end{cases}$$

Since each bonding mapping f_j is monotone and hereditarily quotient, each A_j is connected and $f_j|_{A_{j+1}} : A_{j+1} \longrightarrow A_j$ is monotone and hereditarily quotient. Thus, by Theorem 1.5.3, the inverse limit $\varprojlim (A_j, f_j|_{A_{j+1}}) = \pi_i^{-1}(x_i)$ is connected.

(ii). Let $x \in X$ and U be a neighborhood of x in X. By Lemma 1.5.1 there exists an integer i and an open subset U_i in X_i such that $x \in \pi_i^{-1}(U_i) \subset U$. Then $x_i \in U_i \subset X_i$. Since X_i is locally connected, there exist a connected neighborhood V_i of x_i such that $x_i \in V_i \subset U_i$. $\pi_i^{-1}(V_i)$ is connected by (i) and is a neighborhood of x contained in U as desired.

Theorem 1.5.5 **Anderson-Choquet Embedding Theorem** ([Na1], Theorem 2.10, p.23) *Let $(X,\ d)$ be a compact metric space. Let $\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence where each X_i is a nonempty compact subset of X and each f_i maps X_{i+1} onto X_i. Assume (1) and (2) below:*

(1) For each $\epsilon > 0$ there exists k such that for all $p \in X_k$ diameter$[\bigcup_{j>k} f_{kj}^{-1}(p)] < \epsilon$ and

(2) For each i and each $\delta > 0$ there exists $\delta' > 0$ such that whenever $j > i$ and $p,\ q \in X_j$ such that $d(f_{ij}(p),\ f_{ij}(q)) > \delta$ then $d(p,\ q) > \delta'$.

Then $\varprojlim (X_i, f_i)$ is homeomorphic to $\bigcap_{i=1}^{\infty}(\overline{\bigcup_{m\geq i} X_m})$. In particular, if $X_i \subset X_{i+1}$ for each i then $\varprojlim (X_i, f_i)$ is homeomorphic to $\overline{\bigcup_{i=1}^{\infty} X_i}$.

Let X and Y be metric spaces. A mapping $f : X \longrightarrow Y$ is called an *ϵ-map* provided that f is continuous and the diameter of $f^{-1}(f(x)) < \epsilon$ for all $x \in X$. Let $\mathcal{P}$ be a given collection of metric spaces. Then X is said to be *$\mathcal{P}$-like* provided that for each $\epsilon > 0$ there exists an ϵ-map f_ϵ from X onto some member of $\mathcal{P}$. The union of the simplices (regarded as a subset of $\mathbf{R}^n$ for some positive integer n) belonging to a complex in $\mathbf{R}^n$ forms a closed subset of $\mathbf{R}^n$ and is called a *polyhedron* in $\mathbf{R}^n$.

Theorem 1.5.6 **$\mathcal{P}$-like Theorem** ([Na1], Theorem 2.13, p.24) *If X is a continuum* *and $\mathcal{P}$ is a collection of compact connected polyhedra then X is $\mathcal{P}$-like if and only if X is homeomorphic to $\varprojlim (P_i, f_i)$ where each $P_i \in \mathcal{P}$ and f_i is surjective.*

Chapter 2

Locally Connected Separable Metric Spaces in $D_{\aleph_0}$

In this chapter X denotes a non-degenerate, locally connected , connected, separable metric space in $D_{\aleph_0}$. We show that a locally connected, connected, separable, metric space X with $D^s(X) \leq \aleph_0$ is a rim-countable, hereditarily locally connected, σ-compact ANR which contains only finitely many simple closed curves and finitely many endpoints and, hence, X becomes a R-tree upon removal of finitely many selected points. Conversely, if X is a locally connected, connected, separable, metric space which contains only finitely many simple closed curves and is the union of a R-tree Y with finitely many endpoints and a finite set Z, then X is in $D_{\aleph_0}$. Stone [St] had given another characterization of these spaces. Stone's proof was based on work of Shimrat on D_1-spaces. In the course of obtaining our characterization we abstract properties which allow us to obtain directly Stone's result that every locally connected, connected, separable, metric $D_{\aleph_0}$-space X is a D_n-space for some integer n.

2.1 The Space X is Rim-Countable

Lemma 2.1 *Let $A_0 = \{x \in X : x$ is not a local separating point of $X\}$. Then the set A_0 is finite.*

Proof. Suppose A_0 is infinite, then A_0 contains an infinite relatively discrete subset A_1. Since $D^s(X) \leq \aleph_0$ A_1 separates X. Let us suppose A_1 separates some two points a and b in X. By Lemma 1.1.8, A_1 contains an irreducible subset A_2 separating a and b in X. If $|A_2|$ $= 1$ then $A_2 = \{c\}$ for some $c \in X$. Then c is a separating point of X which is impossible. So $|A_2| \geq 2$. Let $X \setminus A_2 = G \cup H$ where G and H are nonempty separated sets containing the points a and b respectively. Let $d \in cl(G) \cap cl(H)$ and let U be a connected open neighborhood of d such that $U \cap A_2 = \{d\}$. Then $\{d\}$ separates U which is a contradiction since $d \in A_0$. Therefore, A_0 must be finite.

Theorem 2.2 *The space X is σ-compact.*

Proof. Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of X and let $\{U_i\}_{i=1}^{\infty}$ be a countable basis for X with each U_i connected. For each $x \in X \setminus A_0$, by Lemma 1.1.1 there exists an integer k such that $x \in U_k$ and $\{x\}$ disconnects U_k. Since $\bigcup\{a_i\}_{i=1}^{\infty}$ is dense there exist $a_i, a_j \in U_k$ which are separated by x in U_k. Put

$$L_{ij}^k = \{\, x \in U_k : x \text{ separates } a_i \text{ and } a_j \text{ in } U_k\} \cup \{a_i, a_j\}.$$

Since each U_k is connected and locally connected, by Theorem 1.1.9, each L_{ij}^k is a compact, naturally linearly ordered subspace of X. Note that the collection of all such L_{ij}^k's is countable, and their union covers $X \setminus A_0$. Thus, X is σ-compact.

Theorem 2.3 *The space X is rim-countable.*

Proof. From the proof of Theorem 2.2 we have $X = \bigcup_{i=0}^{\infty} A_i$, where A_0 is finite and, for each $i > 0$, A_i is a compact, naturally linearly ordered subspace of X. We then have for each $i \geq 0$ A_i is rim-countable and closed in X. Applying Theorem 1.2.5 X is rim-countable.

Remark The space X may not be rim-finite. Such an example is given in Example 6.2.

2.2 The Space X is Arc Connected

Lemma 2.4 *If U is an open connected subset of X. Then $D^s(U) \leq D^s(X)$.*

Proof. Let $A \subset U$ with $|A| = D^s(X)$. Suppose $U \setminus A$ is connected. Then $cl(U) \setminus A$ is connected. Since X is locally connected, the closure of each component of $X \setminus cl(U)$ meets $cl(U) \setminus A$. We then have that $X \setminus A = (cl(U) \setminus A) \cup (X \setminus cl(U))$ is connected. This is a contradiction and Lemma 2.4 is proved.

By an *open arc* we mean a homeomorphic copy of the open interval (0, 1).

Lemma 2.5 *Let L be an open arc in X and let $x \in L \setminus A_0$. There exists an $\epsilon_x > 0$ such that for any connected open neighborhood U of x in X with $diam(U) \leq \epsilon_x$ x separates in U the two components of $L \cap U$ which have x as a common boundary point.*

Proof. Since x is a local separating point of X there is a connected open neighborhood U_1 of x such that $\text{diam}(U_1) \leq 1$ and x separates U_1. If x does not separate in U_1 the two components r_1 and s_1 of $L \cap U_1$ which have x as a common boundary point, then there exists a finite simple chain $\mathcal{C}_1$ of connected open sets with closures in $U_1 \setminus \{x\}$ from r_1 to s_1. Let U_2 be a connected open neighborhood of x with $U_2 \subset U_1$ and diam $(U_2) \leq \frac{1}{2}d(x, cl(\cup\mathcal{C}_1)) \leq \frac{1}{2}$. Then $\{x\}$ separates U_2. If x does not separate in U_2 the two components r_2 and s_2 of $L \cap U_2$ which have x in their common boundary, then there exists a finite simple chain $\mathcal{C}_2$ of connected open sets with closures in $U_2 \setminus \{x\}$ from r_2 to s_2. This process can be continued. If it stops after finitely many steps, the Lemma will be proved. If the process can be continued through infinitely many steps, we get a decreasing sequence of connected open neighborhoods $\{U_i\}_{i=1}^{\infty}$ of x with $\text{diam}(U_i) \leq \frac{1}{2}d(x, cl(\cup\mathcal{C}_{i-1}) \leq 2^{-i+1}$, a sequence of simple chains $\{\mathcal{C}_i\}_{i=1}^{\infty}$ of connected open sets with closures in $U_i \setminus \{x\}$ from r_i to s_i where r_i and s_i are the components of $L \cap U_i$ with x in their common boundary and $r_{i+1} \subset r_i$ and $s_{i+1} \subset s_i$.

Each $r_i \cup \{x\} \cup s_i \cup (\cup\mathcal{C}_i)$ is connected and no point of the component $int(r_i)$ of x in $r_i \cup s_i \cup \{x\} \setminus cl(\cup\mathcal{C}_i)$ disconnects $r_i \cup \{x\} \cup s_i \cup (\cup\mathcal{C}_i)$. By Lemma 1.1.2, there are only countably many separating points of X in $int(r_i)$. Let $p_1 \in int(r_1) \setminus cl(U_2)$ be a non-separating point of X. If $p_1, ..., p_{i-1}$ have been defined let $p_i \in int(r_i) \setminus cl(U_{i+1})$ be a non-separating point of $X \setminus \{p_1, ..., p_{i-1}\}$. Then $\{p_i\}_{i=1}^{\infty}$ converges to x. But $\bigcup\{p_i\}_{i=1}^{\infty}$ separates X. By Lemma 1.1.8 $\bigcup\{p_i\}_{i=1}^{\infty}$ contains a closed separator of X. Since $lim(p_i) = x$ this closed separator must be finite which is impossible by the construction and Lemma 2.5 is proved.

We recall that a space X is said to have *order n at a point p of X*, denoted by $\text{ord}(p, X) = n$, for some positive integer n provided that X has a neighborhood basis at p of open sets $\{U_\alpha\}$ whose boundaries are exactly n-point sets. The following lemma is a stronger version of Lemma 2.5 .

Lemma 2.6 *If L is an arc in X then there are uncountably many points of L having* *order* 2 *in* X.

Proof. By Lemma 2.5, for $x \in L \setminus A_0$, there exists a rational number $r_x > 0$ such that if U is a connected open neighborhood of x with $\operatorname{diam}(U) \leq r_x$, then $\{x\}$ separates in X the two components of $L \cap U$ which have x as a common boundary point in U. Take r_0 such that $F = \{x \in L : r_x = r_0\}$ is uncountable and take a connected open subset $U_0 \subset X$ such that $\operatorname{diam}(U_0) \leq r_0$ and U_0 contains uncountably many points of F. Each $x \in F \cap U_0$ is a separating point of U_0 and separates in U_0 the two components of $L \cap U$ (which have x as a common boundary point) in U_0. Since $F \cap U_0$ is uncountable, applying Theorem 1.1.10, there exists $Q \subset F \cap U_0$, such that $(F \cap U_0) \setminus Q$ is countable (hence, Q is uncountable) and each $x \in Q$ is of order no more than two in U_0. Since each $x \in Q$ separates U_0 between two points of the component of x in $L \cap U_0$ it follows that x has order 2 in U_0 and, hence, in X as required.

Lemma 2.7 *The space* X *does not contain infinitely many mutually disjoint simple closed curves.*

Proof. Suppose $\{S_i\}_{i=1}^{\infty}$ is a collection of mutually disjoint simple closed curves in X. By Lemma 1.1.2 each S_i contains only countably many separating points of X. Take $p_1 \in S_1 \setminus A_0$ to be a non-separating point of X and let $\epsilon_1 > 0$ as in Lemma 2.5 for p_1, *i.e.*, for each connected open neighborhood U of p_1 in X with $\operatorname{diam}(U) \leq \epsilon_1$, p_1 separates in U the two components of $S_1 \cap U$ which have x in their common boundary. By induction, take $p_{n+1} \in S_{n+1} \setminus (A_0 \cup \{p_1, ..., p_n\})$ to be a non-separating point of $X \setminus \{p_1, ..., p_n\}$, and let $\epsilon_{n+1} > 0$ as in Lemma 2.5 for p_{n+1}. In this manner, we get an infinite sequence of points $\{p_1, p_2, ...\}$. We may assume $\bigcup\{p_i\}_{i=1}^{\infty}$ is a discrete subset of X. For each i, let U_i be a connected open neighborhood of p_i with $\operatorname{diam}(U_i) \leq \epsilon_i$ and $U_i \cap (\bigcup\{p_j\}_{j=1}^{\infty}) = \{p_i\}$. Since $D^s(X) \leq \aleph_0$, $X \setminus \bigcup\{p_i\}_{i=1}^{\infty}$ is the union of two separated sets P and Q. By Lemma 1.1.8 we may assume $\bigcup\{p_i\}_{i=1}^{\infty}$ is an irreducible separator of X with respect to some two points a and b in P and Q respectively, *i.e.*, $bd(P) = bd(Q) = \bigcup\{p_i\}_{i=1}^{\infty}$. Now for each i, $U_i \setminus \bigcup\{p_j\}_{j=1}^{\infty} = U_i \setminus \{p_i\}$ is the union of the separated sets $U_i \cap P$ and $U_i \cap Q$. By the choice of p_i, $S_i \cap U_i \cap P \neq \emptyset$ and $S_i \cap U_i \cap Q \neq \emptyset$. However, $S_i \setminus \bigcup\{p_j\}_{j=1}^{\infty} = S_i \setminus \{p_i\}$ is connected because S_i is a simple closed curve. This is a contradiction and Lemma 2.7 is proved.

Theorem 2.8 *The space* X *contains only finitely many simple closed curves.*

Proof. Suppose $\{S_i\}_{i=1}^{\infty}$ is an infinite sequence of simple closed curves in X. We may suppose for each i $S_{i+1} \not\subset \bigcup_{j=0}^{i} S_j$. By Lemma 2.7 we may suppose there is an i_0 such that S_{i_0} meets infinitely many simple closed curves $\{S_{i_k}\}_{k=1}^{\infty}$ of $\{S_i\}_{i=1}^{\infty}$.

Consider $X_0 = \bigcup_{k=0}^{\infty} S_{i_k}$. Let $C_0 = S_{i_0}$, $x_1 \in S_{i_1} \setminus (S_{i_0} \cup A_0)$, and l_1 the component of $S_{i_1} \setminus S_{i_0}$ containing x_1. Let C_1 be a simple closed curve formed from l_1 and a subarc of C_0. Let $x_2 \in S_{i_2} \setminus (C_0 \cup C_1)$ and let l_2 be the component of $S_{i_2} \setminus (C_0 \cup C_1)$ containing x_2. Since X_0 is not the union of finitely many simple closed curves we continue in the above manner to get a sequence of simple closed curves $\{C_i\}_{i=1}^{\infty}$, open arcs $\{l_i\}_{i=1}^{\infty}$, and points $\{x_i\}_{i=1}^{\infty}$ such that

(*) For all i, $x_i \in l_i \subset C_i$; $l_{i+1} \cap (\bigcup_{j \le i} C_j) = \phi$; $cl(l_{i+1}) \subset l_{i+1} \cup (\bigcup_{j \le i} C_j)$.

Now choose $p_1 \in l_1 \setminus (A_0 \cup (\bigcup_{i=1}^{\infty} (cl(l_i) \setminus l_i)))$ to be a non-separating point of X. By induction, choose $p_{n+1} \in l_{n+1} \setminus (A_0 \cup (\bigcup_{i=1}^{\infty} bd(l_i)))$ to be a non-separating point of $X \setminus \{p_1, ..., p_n\}$ and all the p_n's have the properties in Lemma 2.5. Now if necessary, we could have chosen each C_i more carefully such that $p_j \notin C_i$ for $j < i$ by induction on i. Again with the argument in the proof of Lemma 2.7 we induce a contradiction. This proves Theorem 2.8.

In the following we need to use some results from Whyburn's cyclic element theory (see [Wh1], [Wh2], [Leh]). For the convenience of the reader we state some essential definitions and properties here. For $a, b \in X$ let $L_X(a,b) = \{x \in X : x \textit{ separates } a \textit{ and } b \textit{ in } X\}$ and $E_X(a,b) = L_X(a,b) \cup \{a,b\}$. We say a and b are *conjugate in* X if $L_X(a,b) = \phi$. A subset $E \subset X$ is an E_0*-set* of X if E is non-degenerate, connected, has no separating point of itself, and is maximal with respect to these properties. An *A-set* of X is a closed subset B of X such that $X \setminus B$ is the union of a collection of open sets each bounded by a single point of B. The *cyclic chain in* X *from* a *to* b is $C_X(a,b) = \bigcap\{B : B$ is an A-set of X and $a, b \in B\}$. Then we have the following properties.

a) If B is an A-set of X and if Z is a connected subset of X, then $B \cap Z$ is connected.

b) If a and b are distinct conjugate points of X, then $C_X(a,b)$ is an E_0-set of X.

Theorem 2.9 *The space X is arc connected.*

Proof. We prove first that each arc component of X is closed. Let R be an arc component of X. Suppose $x \in cl(R) \setminus R$. Take $x_i \in R$ such that $\{x_i\}_{i=1}^{\infty}$ converges to x. Since X has only finitely many simple closed curves there are only finitely many arcs from x_i to x_{i+1} for each i. Let $\overline{x_i x_{i+1}}$ denote an arc from x_i to x_{i+1} of minimal diameter in X. We may suppose $d(x, x_{i+2}) \leq \frac{1}{2}d(x, A)$ where A is any arc in X with endpoints x_i and x_{i+1}.

Claim There exists $\epsilon_0 > 0$ *such that* $diam(\overline{x_{i_k} x_{i_{k+1}}}) \geq \epsilon_0$ *for some subsequence* $\{x_{i_k}\}_{k=1}^{\infty}$ *of* $\{x_i\}_{i=1}^{\infty}$.

Proof of Claim. If the claim fails, then $\{diam(\overline{x_i x_{i+1}})\}_{i=1}^{\infty}$ converges to 0. Hence, $\bigcup_{i=1}^{\infty} \overline{x_i x_{i+1}} \cup \{x\}$ is compact, connected and locally connected. It follows that $\bigcup_{i=1}^{\infty} \overline{x_i x_{i+1}} \cup \{x\}$ contains an arc from x_1 to x. This is a contradiction since $x \notin R$ and the claim is proved.

Let U be a connected open neighborhood of x with $\mathrm{diam}(U) \leq min(\epsilon_0, 1)$. We may assume by passing to a subsequence if necessary that $x_k = x_{i_k} \in U$ for all k. So in U there is no arc connecting x_i and x_j for $i \neq j$, *i.e.*, the x_i's belong to distinct arc components of U. Now we consider the subspace U which is still connected, locally connected and $D^s(U) \leq \aleph_0$. Since $E_U(x, x_1)$ is compact but not connected, it has a gap, *i.e.*, there exist two elements a_1 and b_1 of $E_U(x, x_1)$ such that there is no element of $E_U(x, x_1)$ between a_1 and b_1 when $E_U(a_1, b_1)$ is given its natural order from x to x_1. So in U, $E_1 = C_U(a_1, b_1)$ is an E_0-set of U. Pick $p_1 \in E_1$ to be a non-separating point of U. Let $U_1 = U, x_{i_1} = x_1$ and repeat the above argument in $U \setminus \{p_1\}$. Take $U_2 \subset U_1$ to be a connected open neighborhood of x with $\mathrm{diam}(U_2) \leq \frac{1}{2}$ and $p_1 \notin cl(U_2)$. Let $x_{i_2} \in U_2$. Then $E(x, x_{i_2}) \subset U_2$ and $E(x, x_{i_2})$ has a gap, say a_2 and b_2, and so $E_2 = C_{U \setminus \{p_1\}}(a_2, b_2)$ is an E_0-set in $U \setminus \{p_1\}$. Pick $p_2 \in E_2 \cap U_2$ to be a non-separating point of $U \setminus \{p_1\}$. By induction, we get a decreasing sequence of connected open neighborhoods $\{U_i\}_{i=1}^{\infty}$ of x with $\mathrm{diam}(U_i) \leq \frac{1}{i}$, a sequence of points $\{p_i\}_{i=1}^{\infty}$ and a sequence $\{E_i\}_{i=1}^{\infty}$ such that each E_i is an E_0-set of $U \setminus \{p_1, ..., p_{i-1}\}$, $p_i \in U_i \cap E_i$ is a non-separating point of $U \setminus \{p_1, ..., p_{i-1}\}$, and $p_i \notin cl(U_{i+1})$ for each $i > 1$. Therefore, $U \setminus \{p_1, ..., p_i\}$ is connected for each $i \geq 1$. The sequence $\{p_i\}_{i=1}^{\infty}$ converges to x and $\bigcup\{p_i\}_{i=1}^{\infty}$ is a separator of U. By Lemma 1.1.8 $\bigcup\{p_i\}_{i=1}^{\infty}$ contains a finite separator of U. This is impossible by the construction. Therefore, the arc component R is closed.

It remains to show that each arc component of X is open. Let R be an arc component of X and $a \in R$. It suffices to show that a is not a limit point of $X \setminus R$. Otherwise, since arc components are closed , we could pick a sequence $\{a_i\}_{i=1}^{\infty}$ in X converging to a and such that the a_i's belong to distinct arc components of X. Now as in the proof that R is closed and taking $U = X$ we derive a contradiction. Therefore, R is open. Hence, $R = X$ and X is arc connected.

Obviously, the above argument works for any connected open subset of X.

Theorem 2.10 *The space X is locally arc connected.*

As a consequence of Theorem 2.10 and Lemma 2.6 we have the following theorem.

Theorem 2.11 *The set of points of order 2 in X is uncountable and dense in X.*

Lemma 2.12 *If x is a local separating point of the space X which is not a separating point of X then x is contained in a simple closed curve of X.*

Proof. Let U be a connected open neighborhood of x such that $U \setminus \{x\} = V \cup W$, where V and W are two disjoint, nonempty, open sets. Let B be an arc in U which contains one endpoint in V and one in W. Since $X \setminus \{x\}$ is connected there is an arc C in $X \setminus \{x\}$ which meets each of the components of $B \setminus \{x\}$ in exactly one point. Then $B \cup C$ contains a simple closed curve D and $x \in D$.

Theorem 2.13 (Stone [St]) *A locally connected, connected, separable, metric $D_{\aleph_0}$-space X is a D_n-space for some positive integer n.*

Proof. By Lemma 2.1 the set A_0 of all non-local separating points of X is finite. By Theorem 2.8 and Theorem 2.10 the space X contains only finitely many simple closed curves and is locally arc connected. By the above and Lemma 2.5 A_0 is the set of all endpoints of X. Let $\varepsilon(X)$ denote the number of endpoints of X. By Theorem 2.8 and Theorem 2.9 the fundamental group $\pi(X)$ is a free group on finitely many generators. Let $\rho(X)$ be the number of these generators.

We show that X becomes disconnected upon the removal of any set of $\rho(X)+\varepsilon(X)+1$ distinct points: If $\rho(X) = 0$ then X contains no simple closed curve. Let A be a subset of X of cardinality $\varepsilon(X)+1$. Then there is an $x \in A$ which is not an endpoint of X. By Lemma 2.12 x is a separating point of X and, hence, A separates X. Assume Theorem 2.13 is true for locally connected, connected, separable, metric $D_{\aleph_0}$-spaces with $\rho < k$, $k \geq 1$. Let X be a locally connected, connected, separable, metric $D_{\aleph_0}$-space with $\rho(X) = k$ and let A be a subset of X of cardinality $\rho(X) + \varepsilon(X) + 1$. Let $x \in A$ which is not an endpoint of X.

Then x is a local separating point of X. If x is a separating point of X then A separates X. Assume x is not a separating point of X. By Lemma 2.12 x is contained in a simple closed curve of X. Then $X \setminus \{x\}$ is a locally connected, connected, separable, metric $D_{\aleph_0}$-space with $\rho(X \setminus \{x\}) < k$. By the inductive assumption $A \setminus \{x\}$ separates $X \setminus \{x\}$ and, hence, A separates X. Therefore, X is in D_n for $n = \rho(X) + \varepsilon(X) + 1$.

2.3 Characterizations of The Space X

A *R-tree* is a uniquely arc connected, locally arc connected, metric space (see for example [MMOT]). R-trees are 1-dimensional and contractible ARs. An AR is a separable metric space A such that for every separable metric space Y containing A as a closed subspace there is a continuous function $r : Y \longrightarrow A$ such that r restricted to A is the identity. If X is a locally connected, connected, separable metric space with $D^s(X) \leq \aleph_0$ then X becomes a R-tree upon removal of finitely many selected points.

Theorem 2.14 *Let X be a locally connected, connected, separable, metric $D_{\aleph_0}$-space. Then X has finitely many simple closed curves and X is the union of a R-tree with finitely many endpoints and a finite set. Conversely, if X is a locally connected, connected, separable, metric space which contains only finitely many simple closed curves and is the union of a R-tree Y with finitely many endpoints and a finite set Z, then X is in $D_{\aleph_0}$.*

Proof. Let X be a locally connected, connected, separable, metric $D_{\aleph_0}$-space. By Theorem 2.8 X contains at most finitely many simple closed curves. If X contains no simple closed curve then X is a R-tree. Assume Theorem 2.14 holds for all such X which contain no more than n simple closed curves. Now suppose X contains $n+1$ simple closed curves. Let C be a simple closed curve in X. Remove a point x with order 2 (in X) on C by Lemma 2.6. The resulting space $X \setminus \{x\}$ is connected, locally connected, $D^s(X \setminus \{x\}) \leq \aleph_0$ and $X \setminus \{x\}$ contains no more than n simple closed curves. By the hypothesis X becomes a R-tree upon removal of no more than $n+1$ selected points. Hence, X is the union of a R-tree and a finite set. The proof of the converse is clear by the definition of disconnection number.

Stone gave another characterization of the class of locally connected, connected, separable, metric $D_{\aleph_0}$-spaces using Shimrat's characterization of locally connected, connected, separable, metric D_1-spaces. We have given our proof because its arrangement makes clear what is really needed for the proof of Stone's corollary (as Stone had requested). Below we show in Theorem 2.18 that Stone's characterization is equivalent to ours.

Theorem 2.15 **Stone's characterization** [St, Theorem 1]: *Every locally connected, connected, separable, metric $D_{\aleph_0}$-space consists of a connected finite graph L, together with a countable family of pairwise disjoint open ramifications (i.e., locally connected D_1-spaces); these ramifications are open subsets of $X \setminus L$ and the boundary of each in X is a single point of L. Conversely, every such space if it is locally connected, connected, separable and metric then it is in $D_{\aleph_0}$.*

A point p of a space X is called a *branch point of* X provided that $\mathrm{ord}(p,\ X) > 2$.

Lemma 2.16 *The space X has only countably many branch points.*

Proof. Since the space X is the union of a R-tree and a finite set, without loss of generality, we assume X is a separable R-tree. Let B be the set of all branch points of X. Suppose B is uncountable.

Claim *There exist two points a and c in X and an uncountable subset $B_0 \subset B$ such that each $b \in B_0$ separates a and c in X.*

Proof of Claim. Let $\{p_i\}_{i=1}^{\infty}$ be a dense subset of X and let $B_{ij} = \{b \in B : b \text{ separates } p_i \text{ and } p_j\}$ for $i \neq j$. Since each branch point is a separating point in a R-tree, we obtain $B = \bigcup\{B_{ij} : i \neq j\}$. Then there exist i and j such that B_{ij} is uncountable. Let $a = p_i$ and $c = p_j$ and $B_0 = B_{ij}$ as desired in the Claim.

Let A be the only arc from a to c. Then $B_0 \subset A \setminus \{a,\ c\}$. For each $b \in B_0$ we have that $A \setminus \{b\}$ has exactly two components and $X \setminus \{b\}$ has at least three components since $\mathrm{ord}(b,\ X) > 2$. We pick a component R_b of $X \setminus \{b\}$ such that $R_b \cap A = \emptyset$. For $b_1,\ b_2 \in B_0,\ b_1 \neq b_2$. Suppose $x \in R_{b_1} \cap R_{b_2}$. Then one of b_1 and b_2 separates the other two of b_1, b_2 and x, assume b_1 separates x and b_2. This means there exists an arc from x to b_2 through b_1. Then b_2 can not separate x and b_1, or $x \notin R_{b_2}$ which is a contradiction. Hence $R_{b_1} \cap R_{b_2} = \emptyset$ for $b_1 \neq b_2$. It follows that $\{R_b\}_{b \in B_0}$ is an uncountable collection of mutually disjoint open subsets of X. This contradicts that X is a separable metric space. Therefore B must be countable.

Remark. We observe from the proof of Lemma 2.16 that the metrizability in Lemma 2.16 is not necessary. We will use this fact in Chapter 3.

Theorem 2.17 *All save possibly a countable number of points of X are of order 2 in X.*

Proof. The theorem follows from Theorem 2.14, Lemma 2.16 and the fact that X has only finitely many endpoints since $D^s(X) \leq \aleph_0$.

Theorem 2.18 *The following two statements are equivalent.*

(1) X is a locally connected, connected, separable, metric $D_{\aleph_0}$-space which has finitely many simple closed curves and X is the union of a R-tree with finitely many endpoints and a finite set.

(2) X is a locally connected, connected, separable, metric $D_{\aleph_0}$-space consists of a connected finite graph L, together with a countable family of pairwise disjoint open ramifications; these ramifications are open subsets of $X \setminus L$ and the boundary of each in X is a single point of L.

Proof. Let X be a locally connected, connected, separable, metric space which contains only finitely many simple closed curves and X is the union of a R-tree Y with finitely many endpoints and a finite set Z. Let E be the set of endpoints of Y. Let L be the smallest closed connected set in X which contains E and all of the simple closed curves in X. Then L is a finite graph and $X \setminus L \subset Y$. Since Y is a separable R-tree, $X \setminus L$ has only countably many components and each component is open in X and is a R-tree and, hence, a ramification with singleton boundary in L.

Conversely, let X be a locally connected, connected, separable, metric space which consists of a connected finite graph L, together with a countable family of pairwise disjoint open ramifications (*i.e.*, locally connected, D_1-spaces) such that these ramifications are open subsets of $X \setminus L$ and the boundary of each in X is a single point of L. Applying Theorem 2.17 let Z be the smallest set such that $X \setminus Z$ is connected and contains no simple closed curve. Then Z is finite and $Z \subset L$. Each point of $X \setminus Z$ separates $X \setminus Z$ and, hence, $X \setminus Z$ is a R-tree. Therefore, these two statements are equivalent.

2.4 More Properties of The Space X

Theorem 2.19 *The space X is an ANR.*

Proof. From Hanner's Theorem (Theorem 1.3.3) it suffices to note that for each $x \in X$ there exists a open neighborhood U_x of x which is a R-tree. For each $x \in X$ let U_x be a connected open neighborhood of x which contains no simple closed curve. Then U_x is an ANR. Hence, X is an ANR.

Theorem 2.20 *The space X is hereditarily locally connected.*

Proof. From the proof of Theorem 2.19 we know that X is locally a R-tree. For any connected subset A of X and each $x \in A$ let U_x be a small open neighborhood of x in X such that U_x is a R-tree. It suffices to show that $U_x \cap A$ has only finitely many components.

Since A is connected A is also arc connected by Theorem 2.8 and Theorem 2.9. If R_1 and R_2 are two components of $U_x \cap A$, pick two points $a \in R_1$, $b \in R_2$. Then there is an arc L_1 in U_x from a to b and an arc L_2 in A from a to b. Hence $L_1 \cup L_2$ contains a simple closed curve. But we know there are only finitely many simple closed curves in X. Therefore, $U_x \cap A$ has only finitely many components as required.

We call a space X a *hereditarily $D_{\aleph_0}$-space* proved that each connected subspace is a $D_{\aleph_0}$-space.

Theorem 2.21 *Let X be a locally connected, connected, separable, metric, hereditarily $D_{\aleph_0}$-space, then X is a finite graph.*

Proof. Let X be a locally connected, connected, separable, metric, hereditarily $D_{\aleph_0}$-space. By Theorem 2.14, X is the union of a R-tree and a finite set M where each point of M is in a simple closed curve. Without loss of generality we may assume that X is a R-tree. To see that X is a union of finitely many open or closed arcs we suppose the contrary. Then, starting from a fixed point of X, we obtain a closed connected subspace X_0 which is a union of countably many closed arcs such that one of the endpoints of each of these arcs is an endpoint of X_0. This is in contradiction with $D^s(X_0) \leq \aleph_0$.

Chapter 3

$D_{s\omega}$-spaces

We write $X \in D_\omega$ if $X \in D_{\aleph_0}$ and each separator F of X contains a separator of X consisting of finitely many points. We write $X \in D_{s\omega}$ if $X \in D_{\aleph_0}$ and each separator F of X between any two points a and b of X contains a separator of X between a and b consisting of finitely many points. Note that every D_n-space, for some positive integer n, is D_ω and every $D_{s\omega}$-space is D_ω. In this chapter we study the structures of $D_{s\omega}$-spaces.

In Section 3.1 we show that if X is a connected, semi-colocally connected, separable metric $D_{s\omega}$-space, then X is hereditarily locally connected and, hence, X is one of the spaces in Chapter 2.

In Section 3.2 we show that if X is a connected, Hausdorff space in $D_{s\omega}$, then there exists a weaker topology for X which makes X a locally connected, Tychonoff, $D_{s\omega}$-space. Under this weaker topology X satisfies all hypotheses of Theorem 2.14 except (possibly) metrizability.

3.1 $D_{s\omega}$-spaces and Property (*)

We say that a topological space X has property (*) provided that for each connected subset U of X and for each sequence $A_1, A_2, \cdots$ of closed, connected subsets of X each of which meets U and such that $A_i \cap A_j \subset cl(U)$ for each $i \neq j$ we have $Lim\ sup\ A_i \subset cl(U)$ (see [G-T]).

Lemma 3.1 *$D_{s\omega}$-spaces have property (*).*

Proof. Let X be a $D_{s\omega}$-space and let U be a connected subset of X. Let $A_1, A_2, \cdots$ be a sequence of closed, connected subsets of X each of which meets U and such that $A_i \cap A_j \subset cl(U)$ for each $i \neq j$. If there exists $x \in (Lim\ sup\ A_i) \setminus cl(U)$ and let V be a neighborhood of x such that $V \cap cl(U) = \emptyset$. Then, infinitely many A_i meet $Bd(V)$ and the collection $\{V \cap A_i\}$ is pairwise disjoint. Let $p \in U$. Then, $Bd(V)$ separates x and p and, hence, there exists a finite subset B of $Bd(V)$ separating x and p. Let $X \setminus B = P \cup Q$ with P separated from Q, $x \in Q$ and $p \in U \subset P$. This is impossible since infinitely many A_i are disjoint from B and meet both Q and $\{p\}$. Hence, $Lim\ sup\ A_i \subset cl(U)$ as required.

A topological space X is *semi-colocally connected* provided that for each point $x \in X$ and for each neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and $X \setminus V$ has finitely many components. A normal space is said to be *finitely Suslinian* provided it is locally connected and each net $\{A_\alpha\}_{\alpha \in I}$ of distinct, closed, connected, pairwise disjoint subsets of it is *null* (i.e., for every open cover $\mathcal{U}$ of X, there exists $I' \subset I$ such that $I \setminus I'$ is finite and each element of $\{A_\alpha\}_{\alpha \in I'}$ is contained in some element of $\mathcal{U}$.)

Theorem 3.2 *If X is a connected, semi-colocally connected, first countable, normal, T_1, $D_{s\omega}$-space, then X is hereditarily locally connected and, hence, X is finitely Suslinian. In particular, if X is a connected, semi-colocally connected, separable metric $D_{s\omega}$-space then X is the union of a R-tree and a finite set.*

Proof. By Lemma 3.1, X has property (*). By [G-T, Theorem 4.1] X is hereditarily locally connected. By [G-T, Theorem 4.2] X is finitely Suslinian. The last statement now follows by Theorem 2.12.

Remark. Here is a very simple example of a separable metric D_3-space which is not a

$D_{s\omega}$-space. Let $X = \{(x,\ sin(\frac{1}{x})) \in \mathbf{R}^2\colon\ 0 < x < 1\} \cup \{(0,\ 0),\ (0,\ 1)\}$. The infinite set $(\mathbf{R} \times \{\frac{\pi}{4}\}) \cap X$ separates X between $(0,\ 0)$ and $(0,\ 1)$, but no finite set separates X between $(0,\ 0)$ and $(0,\ 1)$. This example is not locally connected. However, it follows from Corollary 2.17 that a locally connected, separable, metric $D_{\aleph_0}$-space is a $D_{s\omega}$-space.

Gladdines' example [Gl] is a hereditarily locally connected, metric $D_{\aleph_0}$-space which is not in $D_{s\omega}$. We present Tymchatyn's description (unpublished) of Gladdines' example. We feel this description is more readable than the original one. Let $C = [0,\ 1) \times [0,\ 1) \cup \{(1,\ 0)\}$. We define a metric d on C: For $(x_1,\ y_1),\ (x_2,\ y_2) \in C \times C$, if $x_1 = x_2$, let $d((x_1,\ y_1),\ (x_2,\ y_2)) = |y_2 - y_1|$; if $x_1 \neq x_2$, let $d((x_1,\ y_1),\ (x_2,\ y_2)) = |x_2 - x_1| + y_1 + y_2$. Then $(C,\ d)$ is a R-tree. Let $\mathbf{N}$ denote the set of natural numbers. Let $\mathbf{N}^\omega = \{A \subset \mathbf{N}\ :\ |A| = \aleph_0\}$. For each $A \in \mathbf{N}^\omega$ let T_A be the quotient space of $C \times A \times \{A\}$ obtained by identifying the set $\{(0,\ 0)\} \times A \times \{A\}$ to a point. Let $X = (\bigoplus_{A \in \mathbf{N}^\omega} T_A)/_\sim$ be the adjunction space where the equivalence relation $\sim$ is defined by $((1,\ 0),\ n,\ \{A\}) \sim ((1,\ 0),\ n,\ \{B\})$ for each $n \in A \cap B$ and $A \neq B \in \mathbf{N}^\omega$. A metric on X is introduced as follows.

Let $P_1 = (x,\ m,\ \{A\}),\ P_2 = (y,\ n,\ \{B\}) \in X$.

(i) If $A = B$ and

(a) $m = n$, let $d(P_1,\ P_2) = d(x,\ y)$;

(b) $m \neq n$, let $d(P_1,\ P_2) = d(x,\ (0,\ 0)) + d((0,\ 0),\ y)$.

(ii) If $A \neq B$ and

(c) $m = n$, let $d(P_1,\ P_2) = d(x,\ (1,\ 0)) + d((1,\ 0),\ y)$;

(d) $m \neq n$, then there exists $C \in \mathbf{N}^\omega \setminus \{A,\ B\}$ such that $m,\ n \in C$ and we then define $d(P_1,\ P_2)$ to be the minimal diameter of arcs which connect P_1 and P_2 in X.

Then $(X,\ d)$ is a metric space in $D_{\aleph_0}$ but not in $D_{s\omega}$. Since for every point x of X there exists a small neighborhood of x which is a R-tree by the construction of X, X is hereditarily locally connected. Here X is necessarily <u>not</u> a separable metric space.

An example of a locally connected continuum not in $D_{\aleph_0}$ in which each separator of it contains a separator consisting of finitely many points may be found in Example 6.2.

The subspace $X = \{(x,\ sin(\frac{1}{x})) \in \mathbf{R}^2\colon\ 0 < x \leq 1\} \cup \{(0,\ 0)\}$ of the plane $\mathbf{R}^2$ is an example of $D_{s\omega}$-space which is not locally connected. But in the following section we will construct a locally connected coarser topology for such a $D_{s\omega}$-space. In particularly, the above space X is an arc in a coarser topology obtained by an order topology (Notice from page 8 that $X = E_X(a,\ b)$ where $a = (0,\ 0)$ and $b = (1,\ sin(1))$).

3.2 Rim-finite Topologies on $D_{s\omega}$-spaces

Let $(X,\ \mathcal{T})$ be a Hausdorff $D_{s\omega}$-space. For an arbitrary point $x \in X$ let $\mathcal{N}_x = \{U \subset X :$ *there exists a point* $y \in X$ *and a separation* $X \setminus F = U \cup V$ *for some finite subset* F *such that* $x \in U$ *and* $y \in V\}$. Then, we have the following properties:

(BP0) For every $x \in X$ each $U \in \mathcal{N}_x$ is open in $(X,\ \mathcal{T})$ and its boundary $Bd(U)$ is finite.

(BP1) For every $x \in X$ $\mathcal{N}_x \neq \emptyset$ and for every $U \in \mathcal{N}_x$, $x \in U$.

(BP2) If $x \in U \in \mathcal{N}_y$ then $U \in \mathcal{N}_x$.

(BP3) For any $U_1,\ U_2 \in \mathcal{N}_x$ there exists $U \in \mathcal{N}_x$ such that $U \subset U_1 \cap U_2$.

Properties (BP0), (BP1) and (BP2) follow directly from the definition of $\mathcal{N}_x$. Property (BP3) also follows from the definition of $\mathcal{N}_x$ because $Bd(U_1 \cap U_2) \subset Bd(U_1) \cup Bd(U_2)$.

Let $\mathcal{F}$ be the collection of all subsets of X that are unions of subcollections of $\bigcup_{x \in X} \mathcal{N}_x$. Then, $\mathcal{F}$ is the topology generated by the neighborhood system $\{\mathcal{N}_x\}_{x \in X}$. Clearly $\mathcal{F}$ is coarser than $\mathcal{T}$. The topological space $(X,\ \mathcal{F})$ is rim-finite (see p.12). Clearly $(X,\ \mathcal{F})$ is still a $D_{s\omega}$-space.

Proposition 3.3 *Every rim-finite Hausdorff space is Tychonoff.*

Proof. Let X be a rim-finite Hausdorff space. Let $x \in X$ and let B be a closed set not containing x. We shall construct a continuous mapping $f :\ X \longrightarrow [0,\ 1]$ such that $f(x) = 0$ and $f(B) = 1$.

Claim 1 *X is regular.*

Proof of Claim 1. For every point $x \in X$ let $\mathcal{N}_x$ be a neighborhood basis of X at x such that each member of $\mathcal{N}_x$ has finite boundary. Let $x \in X$ and let B be a closed set not containing x. Let $U \in \mathcal{N}_x$ such that $U \subset X \setminus B$. For each $y \in Bd(U)$, there exists $U_y \in \mathcal{N}_y$ such that $Bd(U_y)$ separates x and y. Let $V = U \setminus \bigcup_{y \in Bd(U)} cl(U_y)$. Since $Bd(U)$ and each

$Bd(U_y)$ are finite and $Bd(V) \subset \bigcup_{y \in Bd(U)} Bd(U_y)$ we have $V \in \mathcal{N}_x$ and $cl(V) \subset U \subset X \setminus B$ and, hence, X is regular.

Claim 2 *If U and V are two open sets of X such that $cl(U) \subset V$ and U has finite boundary, then there exist two disjoint open sets U_r and V_r with finite boundaries such that*

$$cl(U) \subset U_r \subset V_r^c \subset V.$$

Proof of Claim 2. Since X is regular, for every $y \in Bd(U)$, there exists $U_y \in \mathcal{N}_y$ such that $cl(U_y) \subset V$. Let $U_r = U \cup \bigcup_{y \in Bd(U)} cl(U_y)$. Since $Bd(U)$ and each $Bd(U_y)$ are finite and $Bd(U_r) \subset \bigcup_{y \in Bd(U)} Bd(U_y)$ we have $Bd(U_r)$ is finite and $cl(U_r) \subset V$. Let $V_r = X \setminus cl(U_r)$. We then have $Bd(V_r) = Bd(U_r)$ and $cl(U) \subset U_r \subset cl(U_r) = V_r^c \subset V$ as required.

Now we prove X is Tychonoff: Since X is regular, there exist two disjoint open sets $U_{1/2}$ and $V_{1/2}$ with finite boundaries such that

$$x \in U_{1/2} \subset V_{1/2}^c \subset B^c.$$

Again by regularity there exist two disjoint open sets $U_{1/4}$ and $V_{1/4}$ with finite boundaries such that

$$x \in U_{1/4} \subset V_{1/4}^c \subset U_{1/2}.$$

The set $V_{1/2}^c = cl(X \setminus cl(V_{1/2}))$ has finite boundary, so by Claim 2 there exist disjoint open sets $U_{3/4}$ and $V_{3/4}$ with finite boundaries such that

$$V_{1/2}^c \subset U_{3/4} \subset V_{3/4}^c \subset B^c.$$

Combining the above chains, we have

$$x \in U_{1/4} \subset V_{1/4}^c \subset U_{1/2} \subset V_{1/2}^c \subset U_{3/4} \subset V_{3/4}^c \subset B^c.$$

We can further extend this chain by induction: For any integer m there is a chain

$$x \in U_{1/2^m} \subset V_{1/2^m}^c \subset U_{2/2^m} \subset V_{2/2^m}^c \subset \cdots \subset U_{(2^m-1)/2^m} \subset V_{(2^m-1)/2^m}^c \subset B^c,$$

where $U_{k/2^m}$ and $V_{k/2^m}$ are open sets with finite boundaries for each integer k, $1 \leq k < 2^m$. The construction of this chain results in the following properties:

(i) For each dyadic rational in $[0,\ 1]$, $r = k/2^m, k$ and m integers, there exist disjoint open sets U_r and V_r with finite boundaries such that

$$x \in U_r \subset V_r^c \subset B^c;$$

(ii) For any two dyadic rationals $r_1 < r_2$ we have

$$U_{r_1} \subset V_{r_1}^c \subset U_{r_2} \subset V_{r_2}^c.$$

Henceforth, r and r_1 will denote dyadic rationals in $(0,\ 1)$.

We define our function $f: X \longrightarrow [0, 1]$ by

$$f(x) = \begin{cases} inf\{r: x \in U_r\} & \text{if } x \in \bigcup U_r ; \\ 1 & \text{if } x \notin \bigcup U_r. \end{cases}$$

By our construction, $x \in U_r$ for every dyadic rational r, and if $x \in B$, then $x \notin U_r$ for any r. Thus $f(x) = 0$ and $f(B) = 1$.

To complete the proof we need only show that f is continuous. It is enough to show that $f^{-1}(P)$ is open for P an arbitrary member of a subbasis $\mathcal{B}$ for the topology of $[0, 1]$. Since we are assuming the usual topology for $[0, 1]$, one such subbasis is

$$\{[0, a), (b, 1] : a, b \text{ are irrationals in } [0, 1]\}.$$

We need only show that $f^{-1}[0, a)$ and $f^{-1}(b, 1]$ are open for each irrational a and b in $[0, 1]$. But $f^{-1}[0, a) = \bigcup_{r<a} U_r$ and $f^{-1}(b, 1] = \bigcup_{b<r} V_r$, so both of these sets are open and f is continuous.

Lemma 3.4 *If $(X, \mathcal{T})$ is a separable Hausdorff D_{sw}-space then $(X, \mathcal{F})$ is a separable Tychonoff D_{sw}-space.*

Proof. $(X, \mathcal{F})$ is separable since the identity $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{F})$ is continuous. To complete the proof, by Proposition 3.3, it suffices to show that $(X, \mathcal{F})$ is Hausdorff. Let x and y be two distinct points in X. Since $(X, \mathcal{T})$ is Hausdorff, let W be a neighborhood of x such that $y \notin cl(W)$, i.e., $Bd(W)$ separates x and y in $(X, \mathcal{T})$ and, hence, contains a finite separator F of X between x and y. Let $X \setminus F = U \cup V$ be a separation such that $x \in U$ and $y \in V$. Then, $U \in \mathcal{N}_x$ and $y \notin cl(U)$ in $(X, \mathcal{F})$. This implies that $(X, \mathcal{F})$ is Hausdorff and, hence, $(X, \mathcal{F})$ is Tychonoff by Proposition 3.3.

Lemma 3.5 *If U is an open set with finite boundary in a connected Hausdorff space X, then $cl(U)$ has only finitely many components.*

Proof. Let U be an open set with finite boundary in a connected Hausdorff space X. Just suppose the number of components of $cl(U)$ is infinite. Since $cl(U)$ is not connected, there exists a separation $cl(U) = P_1 \cup P_2$, where P_1 and P_2 are disjoint nonempty closed sets. Note that $Bd(U) = Bd(P_1) \cup Bd(P_2)$ and $Bd(P_1) \cap Bd(P_2) = \emptyset$. If one of P_1 and P_2, say $Bd(P_1)$, is empty, then P_1 will be a closed and open proper subset of X which contradicts the connectedness of X. So both $Bd(P_1)$ and $Bd(P_2)$ are nonempty. One of P_1 and P_2, say P_1, contains infinitely many components of $cl(U)$. We may repeat the above argument for P_1. Since $Bd(U)$ is finite and $|Bd(P_1)| < |Bd(U)|$, continuing in this process at most $|Bd(U)| - 1$ steps we find a nonempty closed and open subset P of $cl(U)$ such that $Bd(P) = \emptyset$. This implies that P is a nonempty proper closed and open subset of X which contradicts the connectivity of X. The proof of the lemma is completed.

Proposition 3.6 *A connected, rim-finite, Hausdorff space is hereditarily locally connected.*

Proof. To prove that a space is locally connected it suffices to prove that components of open sets are open. Let X be a connected, rim-finite, Hausdorff space and let U be an open set of X and $x \in U$. Since X is regular by Proposition 3.3, let V be an open neighborhood of x with finite boundary such that $cl(V) \subset U$. Then, the set $cl(V)$ has only finitely many components by Lemma 3.5. Let $C_1, \cdots, C_m$ be an enumeration of the components of $cl(V)$ and assume $x \in C_1$. Since $x \notin \bigcup_{i=2}^{m} C_i$ and each C_i is closed, $V \setminus \bigcup_{i=2}^{m} C_i$ is an open neighborhood of x contained in C_1 and, hence, C_1 is a connected neighborhood of x. So x is in the interior of the component of U which contains x. Hence, X is locally connected. Note that subspaces of rim-finite spaces are rim-finite. This implies that every connected subspace of X is locally connected since it is rim-finite. Hence, X is hereditarily locally connected.

Combining the above results, we have the following theorem.

Theorem 3.7 *If $(X, \mathcal{T})$ is a non-degenerate, connected, separable, Hausdorff $D_{s\omega}$-space then $(X, \mathcal{F})$ is a hereditarily locally connected (in fact, rim-finite), connected, separable, Tychonoff $D_{s\omega}$-space.*

A *generalized arc* Y is a Hausdorff continuum with exactly two non-separating points. If a and b are the two non-separating points of Y, then $Y = E_Y(a, b)$ (see p.8). Thus a generalized arc Y can be linearly ordered in such a way that the order topology and the original topology coincide. We will denote Y by $[a, b]$. By a *generalized generalized simple closed curve* we mean a Hausdorff continuum which is separated by each of its two points subsets.

Lemma 3.8 *Let X be a non-degenerate, connected, T_1, D_ω-space and let*

$$A_0 = \{x \in X : x \textit{ is not a local separating point of } X\}.$$

Then the set A_0 is finite.

Proof. Suppose A_0 is infinite. Then A_0 contains a countably infinite subset A_1. By our assumption A_1 contains a finite subset A_2 separating X and such that no proper subset of A_2 separates X. If $|A_2| = 1$ then $A_2 = \{c\}$ for some $c \in X$. Then c is a separating point of X which is impossible. So $|A_2| \geq 2$. Let $X \setminus A_2 = G \cup H$ where G and H are nonempty separated sets. Let $d \in cl(G) \cap cl(H)$ and let $U = X \setminus (A_2 \setminus \{d\})$ which is a connected open neighborhood of d such that $U \cap A_2 = \{d\}$. Then $\{d\}$ separates U which is a contradiction since $d \in A_0$. Therefore, A_0 must be finite.

If $(X,\ \mathcal{T})$ is a connected, Hausdorff $D_{s\omega}$-space, then $(X,\ \mathcal{F})$ is a connected, Tychonoff $D_{s\omega}$-space. The set A_0 of all non-locally separating points of $(X,\ \mathcal{F})$ is finite by Lemma 3.8.

Lemma 3.9 *Suppose $(X,\ \mathcal{T})$ is a connected, separable, Hausdorff D_ω-space. Then, the space $(X,\ \mathcal{F})$ does not contain infinitely many mutually disjoint generalized simple closed curves.*

Proof. Below we use the topology of $(X,\ \mathcal{F})$. Just suppose $\{S_i\}_{i=1}^{\infty}$ is a collection of mutually disjoint generalized simple closed curves in X. By Lemma 1.1.2 each S_i contains only countably many separating points of X. Take $p_1 \in S_1 \setminus A_0$ to be a non-separating point of X. Suppose for $i = 1, \cdots, n$ $p_i \in S_i \setminus A_0$ so that $X \setminus \{p_1, ..., p_n\}$ is connected. By induction, take $p_{n+1} \in S_{n+1} \setminus (A_0 \cup \{p_1, ..., p_n\})$ to be a non-separating point of $X \setminus \{p_1, ..., p_n\}$. In this manner, we get an infinite sequence of points $\{p_1, p_2, ...\}$. The set $\bigcup\{p_i\}_{i=1}^{\infty}$ separates X because X is in D_ω and, hence, contains a finite separator of X. This is impossible by the construction and Lemma 3.9 is proved.

Theorem 3.10 *Suppose $(X,\ \mathcal{T})$ is a connected, separable, Hausdorff D_ω-space. Then, the space $(X,\ \mathcal{F})$ contains only finitely many generalized simple closed curves.*

Proof. Suppose $\{S_i\}_{i=1}^{\infty}$ is an infinite sequence of generalized simple closed curves in X. We may suppose for each i, $\bigcup_{j=0}^{i} S_j$ is a finite graph, $S_{i+1} \not\subset \bigcup_{j=0}^{i} S_j$ and, by Lemma 3.9, we may suppose there is an i_0 such that S_{i_0} meets infinitely many generalized simple closed curves $\{S_{i_k}\}_{k=1}^{\infty}$ of $\{S_i\}_{i=1}^{\infty}$.

Consider $X_0 = \bigcup_{k=0}^{\infty} S_{i_k}$. Let $C_0 = S_{i_0}$, $x_1 \in S_{i_1} \setminus (S_{i_0} \cup A_0)$ and let l_1 be the component of $S_{i_1} \setminus S_{i_0}$ containing x_1. Let C_1 be a generalized simple closed curve formed from l_1 and a subarc or a point (if $cl(l_1) = S_{i_1}$) of C_0. Let $x_2 \in S_{i_2} \setminus (C_0 \cup C_1)$ and let l_2 be the component of $S_{i_2} \setminus (C_0 \cup C_1)$ containing x_2. Since X_0 is not the union of finitely many generalized simple closed curves we continue in the above manner to get a sequence of generalized simple closed curves $\{C_i\}_{i=1}^{\infty}$, open arcs $\{l_i\}_{i=1}^{\infty}$ and points $\{x_i\}_{i=1}^{\infty}$ such that

(*) For all i, $x_i \in l_i \subset C_i$; $l_{i+1} \cap (\bigcup_{j \leq i} C_j) = \phi$; $cl(l_{i+1}) \subset l_{i+1} \cup (\bigcup_{j \leq i} C_j)$.

Now choose $p_1 \in l_1 \setminus (A_0 \cup (\bigcup_{i=1}^{\infty}(cl(l_i) \setminus l_i)))$ to be a non-separating point of X. By induction, choose $p_{n+1} \in l_{n+1} \setminus (A_0 \cup (\bigcup_{i=1}^{\infty} bd(l_i)))$ to be a non-separating point of $X \setminus \{p_1, ..., p_n\}$. Now if necessary, we could have chosen each C_i more carefully such that $p_j \notin C_i$ for $j < i$ by induction on i. Again with the argument in the proof of Lemma 3.9 we obtain for each i, $\bigcup_{j=0}^{i} C_j \setminus \{p_1, ..., p_i\}$ is connected which contradicts with that X is in D_ω. This proves Theorem 3.10.

As a consequence of Theorem 3.10 we have the following theorem.

Corollary 3.11 *Every separable Hausdorff D_ω-space contains only finitely many generalized simple closed curves.*

Remark. The separability in Corollary 3.11 is essential. There exists a metric D_1-space containing infinitely many generalized simple closed curves: Let $A = \mathbf{N} \in \mathbf{N}^\omega$ in Gladdines' example (Tymchatyn's description). Let X be the quotient space of $C \times A \times \{A\}$ obtained by identifying the set $\{(0, 0), (1, 0)\} \times A \times \{A\}$ into a point p. Since the quotient mapping is perfect, X is metrizable. Clearly, every point of X separates X and there are infinitely many generalized simple closed curves pass through the point p.

Theorem 3.12 *If $(X, \mathcal{T})$ is a connected Hausdorff $D_{s\omega}$-space, then $(X, \mathcal{F})$ is generalized arc connected and locally generalized arc connected.*

Proof. Since the space $(X, \mathcal{F})$ is Tychonoff and rim-finite, by [Is, Theorem VI.30, p.111], $(X, \mathcal{F})$ has a compactification Y that has a basis $\mathcal{B}$ of open sets whose boundaries are contained in X. By the construction in the proof of [Is, Theorem VI.30] we may assume the boundary of every member of $\mathcal{B}$ is finite and, hence, Y is a hereditarily locally connected continuum since it is rim-finite and $(X, \mathcal{F})$ is connected.

Claim Y *is generalized arc connected and locally generalized arc connected.*

Proof of Claim. It suffices to show that Y is locally generalized arc connected. Let U be a connected open set in Y and $a,\ b \in U$. Let $\mathcal{C}$ be a finite chain of connected open subsets from a to b in U such that $cl(\cup\mathcal{C}) \subset U$. Then $cl(\cup\mathcal{C})$ is a subcontinuum containing a and b. Let Z be an irreducible subcontinuum of $cl(\cup\mathcal{C})$ between a and b. Since Y is hereditarily locally connected, Z is locally connected. For $x \in Z\setminus\{a,\ b\}$, if $Z\setminus\{x\}$ is connected, then we can take a finite chain $\mathcal{D}$ of connected open sets from a to b in $Z\setminus\{x\}$ such that $x \notin cl(\cup\mathcal{D})$ and, hence, $cl(\cup\mathcal{D})$ is a proper subcontinuum of Z containing a and b which contradicts the irreducibility of Z. Therefore, there exist exactly two non-separating points (*i.e.*, a and b) in Z. This implies that Z is a generalized arc from a to b and the Claim is proved.

Now we prove that $(X,\ \mathcal{F})$ is generalized arc connected. Let $a,\ b \in X$ and Z be an arc from a to b in Y. Suppose $z \in Z\setminus X$. We denote $[a,\ z]$ and $[z,\ b]$ be the irreducible arcs in Z from a to z and from z to b respectively and $[a,\ z) = [a,\ z]\setminus\{z\}$, $(z,\ b] = [z,\ b]\setminus\{z\}$. Let $Z_0 = (Y\setminus Z)\cup\{z\}$. Then $Y\setminus Z_0 = Y\setminus[(Y\setminus Z)\cup\{z\}] = Z\setminus\{z\} = [a,\ z)\cup(z,\ b]$ is a separation between a and b. In particularly, $Z_0\cap X$ separates X between a and b and, hence, contains a finite separator F separating a and b in X. By [Is, Theorem VI.39, p.115], F separates a and b in Y, in particular, $z \in F \subset X$. This is a contradiction since z was supposed to be in $Y\setminus X$. Therefore, $Z \subset X$ and, hence, X is generalized arc connected.

Finally we prove that $(X,\ \mathcal{F})$ is locally generalized arc connected. Let U be a connected open set in X and $a,\ b \in U$. The set $\mathrm{Ex}(U) = Y\setminus cl_Y(X\setminus U)$ is open in Y. We claim that $\mathrm{Ex}(U) \subset cl(U)$: For every $x \in \mathrm{Ex}(U) = Y\setminus cl_Y(X\setminus U)$, $x \notin cl_Y(X\setminus U)$. Let V be a neighborhood of x in Y such that $V\cap(X\setminus U) = \emptyset$. But X is dense in Y, so must have $V\cap U \neq \emptyset$ and, hence, $x \in cl(U)$. Further, $X\cap\ \mathrm{Ex}(U) = X\setminus cl_Y(X\setminus U) = X\setminus[X\cap cl_Y(X\setminus U)] = U$. We then have that $\mathrm{Ex}(U)$ is a connected open set in Y containing a and b. Since Y is locally generalized arc connected, there is an arc Z from a to b in $\mathrm{Ex}(U)$. By the above argument we get $Z \subset X$. Hence $Z \subset X\cap\ \mathrm{Ex}(U) = U$. This proves that X is locally generalized arc connected.

We define a *generalized R-tree* to be a uniquely generalized arc connected, locally generalized arc connected, Tychonoff space.

Theorem 3.13 *If $(X, \mathcal{T})$ is a connected, separable, Hausdorff $D_{s\omega}$-space, then $(X, \mathcal{F})$ is the union of a rim-finite generalized R-tree with finitely many endpoints and a finite set.*

Proof. We observed earlier that $(X, \mathcal{F})$ is rim-finite. By Theorem 3.10 $(X, \mathcal{F})$ contains at most finitely many generalized simple closed curves. If $(X, \mathcal{F})$ contains no generalized simple closed curve then $(X, \mathcal{F})$ is a generalized R-tree by Theorem 3.12. Assume Theorem 3.13 holds for all such $(X, \mathcal{F})$ which contain no more than n generalized simple closed curves. Now suppose X contains $n+1$ generalized simple closed curves. Let C be a generalized simple closed curve in X. Remove a non-separating point x (in X) on C by Lemma 1.1.2. The resulting space $X \setminus \{x\}$ is connected, locally connected, $D^s(X \setminus \{x\}) \leq \aleph_0$ and $X \setminus \{x\}$ contains no more than n generalized simple closed curves. By the hypothesis X becomes a generalized R-tree upon removal of no more than $n+1$ selected points. This completes the proof.

Remark. Pierce's example (see Example 6.10 when $W = \mathbb{N}$ the natural numbers) shows that Theorem 3.13 is not always true for D_ω-spaces. In fact, there exists even an example [Ma, Theorem II] of a countable, connected, Hausdorff D_1-space.

Theorem 3.14 *Every separable generalized R-tree in $D_{s\omega}$ is the union of countably many metric arcs.*

Proof. Let X be a generalized R-tree in $D_{s\omega}$. Since the set of endpoints of X is finite, let $\{a_i\}_{i=1}^{\infty}$ be the union of a countable dense set of X and the set of endpoints of X. For every i, j, let A_{ij} be the unique arc from a_i to a_j. For each $x \in X$, if x is an endpoint of x, then $x = a_i$ for some i and, hence, $x \in \bigcup_{i,\, j \in N} A_{ij}$. If x is not an endpoint, then it is a separating point. Let U be a connected open neighborhood of x. Then, there exists a separation $U \setminus \{x\} = U_1 \cup U_2$. Pick $a_i \in U_1$ and $a_j \in U_2$. Then, x separates a_i and a_j in U. This implies that x is on the unique arc from a_i to a_j, or $x \in A_{ij} \subset \bigcup_{i,\, j \in N} A_{ij}$. Hence, $X = \bigcup_{i,\, j \in N} A_{ij}$. To complete the proof we show that each A_{ij} is metrizable. Since each A_{ij} is compact we only need to show that each A_{ij} is separable. Let $A = A_{ij}$ and let D be a countable dense set of X. Let B be the set of all branch points of X. B is countable by the remark of Lemma 2.14. If $A \cap D$ is not dense in A, then for every subarc L of $A \setminus A \cap D$ we show $L \cap B$ is dense in L. Suppose not, then there exists an open subarc $L_0 \subset L \setminus L \cap B$ (without endpoints) such that every point of L_0 has order 2 in X. Hence, L_0 itself is an open subset of X which contradicts with the separability of D. So $L \cap B$ is dense in L for every subarc of $A \setminus A \cap D$. It follows that $A \cap (D \cup B)$ is dense in A and, hence, A is separable as required.

Theorem 3.15 *If X is a non-degenerate, connected, separable, Hausdorff, $D_{s\omega}$-space, then we have $X = \bigcup_{i=0}^{\infty} A_i$, where A_0 is finite and, for each $i > 0$, A_i is a closed linearly ordered set with order topology coarser than the subspace toplogy of X and under the order topology each A_i is a metric arc.*

Proof. Let $(X, \mathcal{T})$ be a connected, separable, Hausdorff, $D_{s\omega}$-space. By Theorem 3.13 $(X, \mathcal{F})$ is the union of a generalized R-tree Y and a finite set Z. By Theorem 3.14, $Y = \bigcup_{i=1}^{\infty} A_i$, where each A_i is a metric arc in $(X, \mathcal{F})$. The inverse image of each A_i under the identity $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{F})$ is a closed linearly order set induced by the topology in $(X, \mathcal{F})$.

Note. Theorem 3.15 is not true for D_{ω}-spaces. Such an example can be found in Example 6.10 when the set W is chosen to be a countable discrete set. Inspired by Theorem 3.15, we ask the following question: If $(X, \mathcal{T})$ is a non-degenerate, connected, separable, Hausdorff, $D_{s\omega}$-space, does there exist a weaker topology $\mathcal{O}$ of X in which $(X, \mathcal{O})$ is generalized arc connected, locally generalized arc connected and metrizable? Actually, it suffices to show that such a $(X, \mathcal{O})$ is first-countable. We note from [C-M] that there exists a nonmetrizable, σ-compact space which is the union of two separable, metrizable, F_σ-subsets. The following result is a partial answer to the question.

Corollary 3.16 *If $(X, \mathcal{T})$ is a non-degenerate, countably compact, connected, separable, Hausdorff, $D_{s\omega}$-space, then the space $(X, \mathcal{F})$ is an generalized arc connected, locally generalized arc connected and metrizable continuum.*

Proof. $(X, \mathcal{F})$ is countably compact since the identity $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{F})$ is continuous. By Theorem 3.15, $(X, \mathcal{F})$ is σ-compact and, hence, $(X, \mathcal{F})$ is compact [Eng, Theorem 3.10.1, p.258]. To complete the proof it suffices to show that $(X, \mathcal{F})$ is metrizable. Since $X = \bigcup_{i=0}^{\infty} A_i$, where A_0 is finite and, for each $i > 0$, A_i is a separable metric arc in $(X, \mathcal{F})$, by [Eng, 4.4.H(a), p.359], $(X, \mathcal{F})$ is metrizable since it is Čech-complete.

Remark. We will see from Theorem 4.15 that the space $(X, \mathcal{F})$ in Corollary 3.16 is actually a metric graph. We still do not know whether $(X, \mathcal{T})$ is compact in Corollary 3.16. We note from [Jo, Theorem 5] that there exists a subspace A of the plane $\mathbf{R}^2$ which is a D_1-space and is not an arc, but there exists a weaker topology on A which makes A an open arc. We will construct, in Example 6.1, a connected separable metric space Z with $D^s(Z) = 1$ (Z is in $D_{s\omega}$) and $\dim(Z) = n$ for any $n \in \{1, 2...., \infty\}$. Hence, in general being an element of $D_{s\omega}$ does not carry an implication concerning the dimension of a space without compactness or local connectedness assumptions.

Chapter 4

Hausdorff Continua in $D_{\aleph_0}$

We recall that a compact and connected space is called a *continuum*. A *generalized arc* is a Hausdorff continuum with exactly two non-separating points. A Hausdorff continuum is called a *generalized graph* if it is a union of finitely many generalized arcs any two of which intersect only in a subset of their endpoints. A generalized arc Y can be linearly ordered in such a way that the order topology and the original topology coincide. We will denote Y by $[a,\ b]$ where a and b are the two non-separating points of Y. In [Na1] Nadler proved that if X is a metric continuum, then $D^s(X) \leq \aleph_0$ if and only if $D^s(X) < \aleph_0$, and, hence, that X is a graph. In this chapter we generalize this theorem to the class of Hausdoff continua. Our proof parallels Nadler's initially but later follows the idea of Chapter 2. A Hausdorff continuum is *indecomposable* if it is non-degenerate and if it is not the union of two of its proper subcontinua. If X is a continuum and $p \in X$, then the set of all $x \in X$ such that $\{p,\ x\}$ is contained in a proper subcontinuum of X is called a *composant* of X. Any two distinct composants of an indecomposable continuum are disjoint. In this chapter, unless stated otherwise, X denotes a non-degenerate Hausdorff continuum with $D^s(X) \leq \aleph_0$.

We are going to use the following two theorems.

Bellamy's Theorem ([Be], Corollary 5) *If X is a non-degenerate indecomposable continuum, then X contains an indecomposable subcontinuum Y with at least $\mathfrak{c}$ composants.*

Gordh's Theorem ([Gor], Theorem 2.7) *If X is a continuum which is irreducible between a pair of points and contains no indecomposable subcontinuum with interior, then* *there exists a monotone continuous map f of X onto a generalized arc such that each point inverse under f has empty interior.*

By using Nadler's method we prove the following Lemma 4.1.

Lemma 4.1 *If Y is a non-degenerate subcontinuum of X, then $D^s(Y) \leq \aleph_0$.*

Proof. Let Y be a proper subcontinuum of X, and let $A \subset Y$ with $|A| = \aleph_0$. Suppose that $Y \setminus A$ is connected.

Claim *The number of components of $X \setminus Y$ is finite.*

Proof of Claim. If not, we could choose infinitely many components, $\{C_i\}_{i=1}^{\infty}$, of $X \setminus Y$. Since $C_i \cup Y$ is a continuum for each i, by the Non-Separating Point Existence Theorem (Theorem 1.1.4) and Corollary 1.1.5, no proper connected subset of $C_i \cup Y$ contains the set of all non-separating points of $C_i \cup Y$. For each i let p_i be a non-separating point of $C_i \cup Y$ such that $p_i \in C_i$. Hence

$X \setminus \{p_i\}_{i=1}^{\infty} = \bigcup_{i=1}^{\infty}[(C_i \cup Y) \setminus p_i] \cup \bigcup\{C :$ C is a component which is different from that of C_i's $\}$ is connected. This contradicts that $D^s(X) \leq \aleph_0$ and the claim is proved.

Let $C_1, \cdots, C_m$ be all components of $X \setminus Y$. We pick $q_i \in cl(C_i) \cap Y$ for each $1 \leq i \leq m$. Since $Y \setminus A \subset (Y \setminus A) \cup \{q_1, \cdots, q_m\} \subset Y = cl(Y \setminus A)$, $(Y \setminus A) \cup \{q_1, \cdots, q_m\}$ is connected. Hence

$$X \setminus (A \setminus \{q_1, \cdots, q_m\}) = \bigcup_{i=1}^{\infty}(C_i \cup \{q_i\}) \bigcup (Y \setminus A) \cup \{q_1, \cdots, q_m\}$$

is connected. This contradicts that $D^s(X) \leq \aleph_0$ and Lemma 4.1 is proved.

Lemma 4.2 *The space X is hereditarily decomposable.*

Proof. If there exists an indecomposable subcontinuum Y in X, by Bellamy's theorem, Y contains an indecomposable subcontinuum Z with at least c composants. By Lemma 4.1, $D^s(Z) \leq \aleph_0$. So for any countable subset $A \subset Z$ there exists a composant C of Z missing A. But C is dense in Z, so $Z \setminus A$ is connected. This is contrary to $D^s(Z) \leq \aleph_0$ and the lemma is proved.

Lemma 4.3 *If Y is a subcontinuum of X which is irreducible between a pair of points, then Y is a generalized arc.*

Proof. By Lemma 4.1 and Lemma 4.2 we know that $D^s(Y) \leq \aleph_0$ and Y is a hereditarily decomposable continuum. Using Gordh's theorem, let f be a monotone continuous map from Y onto a generalized arc $[a,\ b]$ with a and b two non-separating points of $[a,\ b]$ such that $\mathrm{Int}(f^{-1}(t)) = \phi$ for each $t \in [a,\ b]$. We only need to show that for each $t \in [a,\ b]$ $f^{-1}(t)$ is a singleton. If not, there exists a $t_0 \in [a,\ b]$ such that $f^{-1}(t_0)$ is non-degenerate and connected and, hence, uncountable. If $t_0 = a$ (or $t_0 = b$) then $f^{-1}(a,\ b]$ (or $f^{-1}[a,\ b)$) is a connected dense subset in Y since f is monotone and $\mathrm{Int}(f^{-1}(t)) = \phi$ for each $t \in [a,\ b]$. Hence, if A is an infinite subset of $f^{-1}(t_0)$, the subset $Y \setminus A$ is still connected. This is contrary to $D^s(Y) \leq \aleph_0$. If $a < t_0 < b$ then $(cl(f^{-1}[a,\ t_0)) \cap f^{-1}(t_0)) \bigcup (cl(f^{-1}(t_0,\ b]) \cap f^{-1}(t_0)) = f^{-1}(t_0)$ since $\mathrm{Int}(f^{-1}(t_0)) = \phi$. Without loss of generality we assume $cl(f^{-1}[a,\ t_0)) \cap f^{-1}(t_0)$ is infinite. Let B be an infinite subset of $cl(f^{-1}[a,\ t_0)) \cap f^{-1}(t_0)$. Since $cl(f^{-1}[a,\ t_0))$ is a subcontinuum of Y with $f^{-1}[a,\ t_0)$ as a connected dense subset, the subset $cl(f^{-1}[a,\ t_0)) \setminus B$ is still connected. This is contrary to Lemma 4.1. This completes the proof of Lemma 4.3.

Corollary 4.4 *Every non-degenerate subcontinuum of X is generalized arc connected.*

Theorem 4.5 *The space X is hereditarily locally connected.*

Proof. If not, by Theorem 1.4.1, there exists a convergence continuum K with a net of continua $\{K_\lambda\}_{\lambda \in \Lambda}$ such that $Lim\ K_\lambda = K$, $K_{\lambda'} \cap K_\lambda = K_\lambda$ or $K_{\lambda'} \cap K_\lambda = \phi$ for $\lambda', \lambda \in \Lambda$ and $K_\lambda \cap K = \phi$ for each λ. Since K is non-degenerate, by Lemma 4.2, $K = A \cup B$ where A and B are two proper subcontinua of K. By Corollary 4.4, for each $\lambda \in \Lambda$, let L_λ be an irreducible generalized arc from K_λ to a point a_λ of K such that $L_\lambda \cap K = \{a_\lambda\}$. Since $\bigcup\{a_\lambda\}_{\lambda \in \Lambda} \subset A \cup B$, either A or B contains a cofinal subset of $\bigcup\{a_\lambda\}_{\lambda \in \Lambda}$. We assume by passing to a cofinal subset if necessary that $\bigcup\{a_\lambda\}_{\lambda \in \Lambda} \subset A$. Then $Y = cl(K \cup \bigcup_{\lambda \in \Lambda} K_\lambda \cup \bigcup_{\lambda \in \Lambda} L_\lambda)$ is a subcontinuum of X with $A \cup \bigcup_{\lambda \in \Lambda} K_\lambda \cup \bigcup_{\lambda \in \Lambda} L_\lambda$ connected and dense in Y. Let $C \subset B \setminus A$ be a countably infinite subset. Then $Y \setminus C$ is connected. This is contrary to $D^s(Y) \leq \aleph_0$ and Theorem 4.5 is proved.

Lemma 4.6 *If U is a connected open set in X then $Bd(U)$ is finite.*

Proof. Suppose $Bd(U)$ is infinite. Let A be a countable infinite subset of $Bd(U)$. Since $U \subset cl(U) \setminus A \subset cl(U)$, $cl(U) \setminus A$ is connected which contradicts with $D^s(cl(U)) \leq \aleph_0$ by Lemma 4.1. Therefore, $Bd(U)$ is finite.

Combining Theorem 4.5 and Lemma 4.6, we have

Corollary 4.7 *The space X is a rim-finite space and, hence, a $D_{s\omega}$-space.*

Lemma 4.8 *If Y and Z are generalized graphs such that $Y \cap Z$ is nonempty and finite then $Y \cup Z$ is a generalized graph.*

Proof. The proof is clear.

For a given integer $n \geq 3$ a *generalized simple n-od A* is the union of n generalized arcs $A_1, \ldots, A_n$ such that there exists a point $p \in A$ with $A_i \cap A_j = \{p\}$ for $i \neq j$ and p is an endpoint of each of A_i and A_j. The point p is called the *vertex* of A. When $n = 3$ we say *A is a generalized simple triod.*

Lemma 4.9 *If the space X contains no generalized simple triod, then X is a generalized arc or a generalized simple closed curve.*

Proof. Let p and q be two non-separating points of X. Let A be a generalized arc in X with endpoints p and q. Since $X \setminus \{p\}$ is open and connected, by Theorem 4.5, it is generalized arc connected. Suppose X contains no generalized simple closed curve. Then X is uniquely arc connected and locally arc connected. Let a and b be two non-separating points of X. Since X contains no generalized simple triod, $X = [a, b]$, an arc. Now suppose X contains a generalized simple closed curve S. Since X is generalized arc connected and contains no generalized simple triod, $X = S$ as required.

Corollary 4.10 *Let Y is a locally connected continuum. For each $x \in Y$ ord$(x, Y) \leq 2$ if and only if Y is a generalized arc or a generalized simple closed curve.*

Lemma 4.11 *Let $p \in X$ such that ord$(p, X) = n < \aleph_0$. Then there exists a local base $\{B_\lambda\}_{\lambda \in \Lambda}$ at p such that each B_λ is an open and connected subset of X and $|bd(B_\lambda)| = n$.*

Proof. Let $\{U_\gamma\}_{\gamma \in \Gamma}$ be a local base at p such that each U_γ is open and $|bd(U_\gamma)| = n$. For each $\gamma \in \Gamma$ let V_γ be the component of p in U_γ. Since X is locally connected each V_γ is open. Also, $bd(V_\gamma) \subset bd(U_\gamma)$ and $\mathcal{V} = \{V_\gamma\}_{\gamma \in \Gamma}$ is a local base at p. Hence, $\mathcal{B} = \{B \in \mathcal{V} : |bd(B)| = n\}$ will be a local base at p with the required property.

Lemma 4.12 *Suppose the space X has only one point p of order ≥ 3 and ord$(p, X) = n < \aleph_0$. Then p is the vertex of a generalized simple n-od which is a neighborhood of p in X.*

Proof. We use the idea in the proof of [Na1, Lemma 9.9]. By Lemma 4.11 let $\mathcal{B} = \{B_\lambda\}_{\lambda \in \Lambda}$ be a local base at p such that each B_λ is an open and connected subset of X and $|bd(B_\lambda)| = n$. If for each $\lambda \in \Lambda$ there exists $x_\lambda \in bd(B_\lambda)$ such that x_λ is not a limit point of $X \setminus B_\lambda$ then $\mathcal{B}' = \{B_\lambda \cup \{x_\lambda\}\}$ forms a local base at p such that $|bd(B_\lambda \cup \{x_\lambda\})| = n - 1$ which contradicts that ord$(p, X) = n < \aleph_0$. Hence there exists $\lambda_0 \in \Lambda$ such that for each

$p_i \in bd(B_{\lambda_0})$, $1 \leq i \leq n$, is a limit point of $X \setminus B_{\lambda_0}$. Note that $cl(B_{\lambda_0})$ is arc connected and locally arc connected (Corollary 4.4) and $\text{ord}(x,\ X) = 2$ for all $x \neq p$ in $cl(B_{\lambda_0})$. It follows that each p_i must be an end point of any arc in $cl(B_{\lambda_0})$ to which p_i belongs. Let $A_i \subset cl(B_{\lambda_0})$ be an arc with endpoints p and p_i such that $A_i \cap A_j = \{p\}$ for $i \neq j$. Then $\bigcup_{i=1}^{m} A_i$ is a generalized n-od with vertex p. Since $\text{ord}(p,\ X) = n$ it follows that $cl(B_{\lambda_0}) = \bigcup_{i=1}^{m} A_i$ as required.

Theorem 4.13 *A Hausdorff continuum X is a generalized graph if and only if $D^s(X) \leq \aleph_0$ and $ord(x,\ X) \leq 2$ for all but finitely many $x \in X$.*

Proof. The necessity is clear. To prove sufficiency let X be a Hausdorff continuum such that $D^s(X) \leq \aleph_0$ and $\text{ord}(x,\ X) \leq 2$ for all but finitely many $x \in X$. By Corollary 4.7, $\text{ord}(x,\ X) < \aleph_0$ for all $x \in X$. If no points are of order ≥ 3 in X then, applying Corollary 4.10, X is a generalized graph. We assume inductively that Theorem 4.13 holds for all continua with at most n points of order ≥ 3. Now suppose X has exactly $n+1$ points, $\{p_i\}_{i=1}^{n+1}$, of order ≥ 3. Since X is locally connected let U be a connected open neighborhood of p_1 such that $p_i \notin cl(U)$ for any $i \geq 2$. In $cl(U)$, p_1 is the only point of order ≥ 3. Let $\text{ord}(p_1,\ cl(U)) = n$. Applying Lemma 4.12 let V be a connected open neighborhood of p_1 in $cl(U)$ such that $cl(V)$ is a generalized n-od. Since $|bd(V)| = n$, $X \setminus V$ has at most n components, $K_1, \cdots, K_m$ $(m \leq n)$. Since $p_1 \notin K_i$ for each $i \geq 1$ by the inductive assumption each K_i is a generalized graph. Note that $\emptyset \neq K_i \cap cl(V) \subset bd(V)$ and $(cl(V) \cup K_i) \cap K_j = cl(V) \cup K_j$ for $i \neq j$. By Lemma 4.8 $K_i \cup cl(V)$ is a graph for each i and hence $X = cl(V) \cup \bigcup_{i=1}^{m} K_i$ is a generalized graph. This completes the proof of Theorem 4.13.

Lemma 4.14 *Let X be a Hausdorff continuum with $D^s(X) \leq \aleph_0$ then $ord(x,\ X) \leq 2$ for all but finitely many $x \in X$.*

Proof. Suppose there exists an infinite subset C of X such that for each $x \in C$ $\text{ord}(x,\ X) \geq 3$. Without loss of generality, we assume the set C is countable and contains no cluster point of itself. We shall define a subcontinuum L of X such that the set of endpoints of L is infinite which is contrary to $D^s(L) \leq \aleph_0$, and, hence, completes the proof.

If there exists a generalized arc A such that A contains an infinite subset $\{x_1, ..., x_n, ...\}$ of C. Since for each i, $\text{ord}(x_i,\ X) \geq 3$ and $\text{ord}(x_i,\ A) \leq 2$, let U_i be an open neighborhood of x_i and $p_i \in U_i \setminus A$ such that $U_i \cap U_j = \phi$ for $i \neq j$ and let L_i be a generalized arc in U_i with endpoints x_i and p_i. Then $L = cl(A \cup \bigcup_{i=1}^{\infty} L_i)$ is a subcontinuum with $\bigcup_{i=1}^{\infty}\{p_i\}$ in its set of endpoints.

We assume that no generalized arc contains infinitely many points of C. Let x_0 be a limit point of C. Let U_1 be a connected open neighborhood of x_0 and take $x_1 \in U_1 \cap C$. Let L_1 be a generalized arc in U_1 from x_1 to x_0. By induction, suppose we have defined $x_1, ..., x_n$, $U_1, ..., U_n$ and $L_1, ..., L_n$ such that each U_i is a connected open neighborhood of x, $cl(U_{i+1}) \subset U_i$, L_i is a generalized arc in U_i from x_i to x_0 and $x_j \notin cl(U_i)$ for $j < i$. Let U_{n+1} be a connected open neighborhood of x_0 such that $cl(U_{n+1}) \subset U_n$ and $x_i \notin cl(U_{n+1})$ for each $i \leq n$. Take $x_{n+1} \in U_{n+1} \cap C \setminus \bigcup_{i=1}^{n} L_i$ and let L_{n+1} be a generalized arc in U_{n+1} from x_{n+1} to x_0. With this construction we have that for each i, $x_i \notin cl(\bigcup_{j \neq i} L_j)$. Then the subcontinuum $L = cl(\bigcup_{i=1}^{\infty} L_i)$ has $\{x_i\}_{i=1}^{\infty}$ contained in its set of endpoints as required.

Theorem 4.15 *A nondegenerate, Hausdorff continuum X is a generalized graph if and only if $D^s(X) \leq \aleph_0$.*

Proof. The theorem follows from Theorem 4.13 and Lemma 4.14.

We recall from Chapter 2 that, since the space X contains only finitely many simple closed curves by Theorem 4.15, there exists the smallest nonnegative integer m, denoted by $\rho(X)$, such that if we remove some m points X becomes a generalized R-tree. Let $\varepsilon(X)$ denotes the number of endpoints of X which is finite. We then have the following corollary from Theorem 4.15.

Corollary 4.16 *Let X be a nondegenerate, Hausdorff continuum with $X \in D_{\aleph_0}$. There is a positive integer n such that $D^s(X) \leq n$. In fact, $D^s(X) = \rho(X) + \varepsilon(X) + 1$.*

Chapter 5

The Connectivity Degrees of Spaces

Let X be a topological space and let a and b be two points of X. A subset of X is said to *join* a and b if a and b are contained in the closure of some component of the set. The space X is said to be *n-point connected between a and b* if no subset of X with fewer then n-points separates a and b in X. We say there exist κ *independent connections between a and b in* X if there exist κ disjoint open sets in X which join a and b (see [Wh3] and [Tym]). We define the *connectivity degree*, $C_m(X)$, of X by $C_m(X) = sup\{\ \kappa$: *there exist two points* a *and* b *in* X *with* κ *independent connections between* a *and* $b\}$. In this chapter we begin to study the relations between connectivity degree and disconnection number.

We are going to use the following theorem.

The n-Open Connections Theorem ([Tym], Theorem 1) *The locally connected, regular, T_1 space X is n-point connected between two points a and b if and only if there exist n disjoint open sets in X which join a and b.*

Corollary 5.1 *If X is a hereditarily locally connected, locally arc connected, connected, metric space that is n-point connected between two points a and b, then X contains n disjoint open arcs joining a and b.*

Proof. By the n-Open Connections Theorem there exist n disjoint open sets $U_1, \cdots, U_n$

in X which join a and b. Since X is locally arc connected and U_i is open for each i we may suppose U_i is connected and locally arc connected for each i. For each i let $c_i \in U_i$ and let $\{x_{ij}\}_{j=1}^{\infty}$ be a sequence in U_i converging to a. Inductively, we construct for each j an arc $c_i x_{ij}$ from c_i to x_{ij} such that for each n, $\bigcup_{j=1}^{n} c_i x_{ij}$ is a tree. Since $U_i \cup \{a\}$ is connected and locally connected we may suppose $\lim_{n\to\infty}(\bigcup_{j=1}^{n+1} c_i x_{ij} \setminus \bigcup_{j=1}^{n} c_i x_{ij}) = \{a\}$. Then $cl(\bigcup_{j=1}^{\infty} c_i x_{ij}) = \bigcup_{j=1}^{\infty} c_i x_{ij} \cup \{a\}$ is a compact tree. So there is an arc in $U_i \cup \{a\}$ from c_i to a. Similarly, there is an arc in $U_i \cup \{b\}$ from c_i to b. Hence, there is an open arc in U_i which joins a and b. Therefore, X contains n disjoint open arcs joining a and b.

Theorem 5.2 *If X is a locally connected and connected separable metric space with $D^s(X) \leq \aleph_0$ then X has finite connectivity degree.*

Proof. Let X be a locally connected and connected separable metric space with $D^s(X) \leq \aleph_0$. By Theorem 2.8 X contains only finitely many simple closed curves. Let k be the number of simple closed curves in X. Then there exist at most $k+1$ independent arcs between any pair of points (the interiors of these arcs are mutually disjoint). By Theorem 2.10, X is a locally arc connected. Therefore, by Corollary 5.1, we have $C_m(X) \leq k+1$.

Theorem 5.3 *If X is a locally connected and connected separable metric space with finite connectivity degree then every two points of X can be separated by a finite subset of X.*

Proof. Since $C_m(X) = k$ for some positive integer k for any pair of points a and b in X there do not exist $k+1$ independent connections between a and b in X. By Corollary 5.1 again X is not $(k+1)$-point connected between a and b. So there exists a subset of X with fewer than $(k+1)$ points and which separates a and b.

Theorem 5.4 *If X is a locally connected and connected separable metric space with $D^s(X) \leq \aleph_0$ then $C_m(X) \leq D^s(X)$.*

Proof. Let X be a locally connected and connected separable metric space with $D^s(X) \leq \aleph_0$. By Corollary 2.19 $D^s(X) = n$ for some positive integer n. Let a and b be two points of X. Suppose there exist κ independent arcs $A_1, \cdots, A_\kappa$ from a to b. For each arc A_i we pick an interior point p_i in A_i of order 2 (Lemma 2.6) in X. Let $A = \{p_1, \cdots, p_\kappa\}$. Then we must have $\kappa = |A| \leq n$. Therefore $C_m(X) \leq D^s(X)$.

With analogous arguments we have the following two theorems.

Theorem 5.5 *If X is a Hausdorff continuum with $D^s(X) \leq \aleph_0$ then X has finite connectivity degree.*

Proof. Let X be a Hausdorff continuum with $D^s(X) \leq \aleph_0$. By Theorem 4.13 X is a generalized graph. Hence X has only finitely many simple closed curves. Let k be the number of simple closed curves in X. Then there exist at most $k+1$ independent arcs between any pair of points of X. Therefore, $C_m(X) \leq k+1$.

Theorem 5.6 *If X is a Hausdorff continuum with $D^s(X) \leq \aleph_0$ then $C_m(X) \leq D^s(X)$.*

Proof. Let X be a Hausdorff continuum with $D^s(X) \leq \aleph_0$. By Corollary 4.14 $D^s(X) = n$ for some positive integer n. Let a and b be two points of X. Suppose there exist κ independent arcs from a to b. For each arc we pick an interior point of order 2 (Lemma 4.12). Let A be the set of those points. Then no proper subset of A disconnects X. Thus $|A| \leq n$. Therefore, $C_m(X) \leq D^s(X)$.

We define a continuum X a *Θ-continuum of type n* for some positive integer n provided there exist two points a and b in X such that $X = \bigcup_{i=1}^{n} A_i$ where each A_i is an arc and $A_i \cap A_j = \{a, b\}$ for $i \neq j$. Let (X, ρ) and (Y, d) be compact metric spaces. A continuous surjection $f : X \longrightarrow Y$ is called a *near homeomorphism* provided that for any $\epsilon > 0$ there is a homeomorphism $h : X \longrightarrow Y$ such that $sup_{x \in X} d(f(x), h(x)) < \epsilon$.

Theorem 5.7 *Let $X = \varprojlim (X_i, f_i)$ where each X_i is a locally connected Θ-continuum of type n and each bonding mapping f_i is a monotone surjection. Then X is also a Θ-continuum of type n.*

Proof. It is easy to see that a monotone mapping from a Θ-continuum of type n onto a Θ-continuum of type n is a near homeomorphism. Hence, Theorem 5.7 is a direct corollary of Brown's Theorem [Bro, Theorem 4].

Theorem 5.8 *Let $X = \varprojlim (X_i, f_i)$ where each X_i is a locally connected continuum and each bonding mapping f_i is an open, monotone surjection.*

Then $C_m(X) \geq sup\{C_m(X_i)\}_{i=1}^{\infty}$.

Proof. For a fixed i let $C_m(X_i) = n$ where n maybe infinite. Let a and b be two points in X such that there exist n independent connections between $\pi_i(a)$ and $\pi_i(b)$ in X_i. Let $U_1, \cdots, U_n$ be such n independent connections. Since the bonding mappings are open, monotone and surjection, the i-th projection π_i is also open, monotone and surjection by [Pu, Theorem 5]. Since a, $b \in cl(\pi_i^{-1}(U_j)) = \pi_i^{-1}(cl(U_j))$ for each j, $\pi_i^{-1}(U_1), \cdots, \pi_i^{-1}(U_n)$ are n independent connections between a and b in X. Therefore $C_m(X) \geq C_m(X_i)$ for each i and, hence, $C_m(X) \geq sup\{C_m(X_i)\}_{i=1}^{\infty}$.

Remark. In Chapter 6 we will give several examples to show how inverse limits affect connectivity degree and disconnection number. Theorem 5.3 fails for non-locally connected spaces (Example 6.12) and this example also gives a negative answer to a question in [Tym]. The following question is still open: Could we improve the inequality in Theorem 5.8 to be an equality by applying Theorem 5.7?

Chapter 6

Examples and Questions

In this chapter we give some examples around the theory we have established in the previous chapters. We show that for any $n \in \{1, 2...., \infty\}$ there is a connected separable metric space Z with $D^s(Z) = 1$ and $\dim(Z) = n$ (Example 6.1). Hence, in general being an element of $D_{s\omega}$ does not carry an implication concerning the dimension of a space. We give an example of a locally connected, connected, separable metric space X with $D^s(X) = 1$ such that X is not rim-finite (Example 6.2). This example also show that the disconnection numbers are not monotone: there exists a closed connected subset Y of X such that $D^s(X) = 1$ and $D^s(Y)$ is not defined. Inverse limits affect disconnection numbers and connectivity degrees of spaces (Examples 6.6 - 6.9). Disconnection number and connectivity degree are different (Examples 6.10 - 6.11). The n-open connections theorem fails for non-locally connected spaces (Example 6.12) and this example is also a negative answer to a question in [Tym].

Example 6.1 *For each $n \in \{1, 2, ..., \infty\}$ there exists a connected separable metric space Z with $D^s(Z) = 1$ and $dim(Z) = n$.*

The example is based on a construction of Lelek ([Lel]). We construct it by the following steps. Let T be the Cantor ternary set in $[0, 1]$. Let $\Delta = T \setminus \{0, 1\}$. For any interval $(a, b) \subset (0, 1)$ let $\Delta(a, b)$ be the image of Δ under the linear homeomorphism from $[0, 1]$ onto $[a, b]$. We call $\Delta(a, b)$ the *basic Cantor set* in (a, b).

Step 1. Let n be a positive integer. In the $(n+1)$-cube $I^{n+1} = \prod\{I_k : k = 1, ..., n+1\}$ where each $I_k = [0, 1]$, let $\pi_i : I^{n+1} \longrightarrow I_i$ denote the i-th coordinate projection. Let $\pi = \pi_1$

and let $A = \pi^{-1}(0), B = \pi^{-1}(1)$. Let $\mathcal{C}$ be the collection of all subcontinua in I^{n+1} meeting both A and B. Then $\mathcal{C}$ has cardinality c. Let $\alpha : \Delta \longrightarrow \mathcal{C}$ be a 1-1 correspondence. For each $t \in \Delta$ let $y_t \in \pi^{-1}(t) \cap \alpha(t)$ and put $Y = \{y_t : t \in \Delta\}$. Then (see [Lel]) Y is totally disconnected and $dim(Y) = n$.

Step 2. Let $\Delta_0 = \Delta$, $\mathcal{C}_0 = \mathcal{C}$, $\alpha_0 = \alpha$ and $Y_0 = Y$. Let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be the sequence of complementary components of Δ in $(0, 1)$. For every (a_i, b_i), let $\Delta(a_i, b_i)$ be the basic Cantor set in (a_i, b_i). Let $\mathcal{C}_1^i$ be the collection of all subcontinua in I^{n+1} meeting both $\pi^{-1}(a_i)$ and $\pi^{-1}(b_i)$. Then $\mathcal{C}_1^i$ has cardinality c. Let $\mathcal{C}_1 = \bigcup_{i=1}^{\infty} \mathcal{C}_1^i$. Let $\Delta_1 = \bigcup_{i=1}^{\infty} \Delta(a_i, b_i)$ and let $\alpha_1 : \Delta_1 \longrightarrow \mathcal{C}_1$ be a function such that $\alpha_1|_{\Delta(a_i, b_i)} : \Delta(a_i, b_i) \longrightarrow \mathcal{C}_1^i$ is a 1-1 correspondence for each i. For each $t \in \Delta_1$ let $y_t \in \pi^{-1}(t) \cap \alpha_1(t)$ and put $Y_1 = \{y_t : t \in \Delta_1\}$. Let $\{(a_{ij}, b_{ij})\}_{j=1}^{\infty}$ be the complementary components of $\Delta(a_i, b_i)$ in (a_i, b_i) for every i.

Step 3. Inductively, we define sequences $\{\Delta_k\}_{k=0}^{\infty}$, $\{\mathcal{C}_k\}_{k=0}^{\infty}$, $\{\alpha_k\}_{k=0}^{\infty}$ and $\{Y_k\}_{k=0}^{\infty}$ satisfying the following conditions:

For each $k \geq 2$,

(a) $\Delta_k = \bigcup_{i_1, i_2, \cdots, i_k = 1}^{\infty} \Delta(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k})$, where each $\Delta(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k})$ is the basic Cantor set in $(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k})$ and $\{(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k})\}_{i_k=1}^{\infty}$ is the sequence of complementary components of $\Delta(a_{i_1, i_2, \cdots, i_{k-1}}, b_{i_1, i_2, \cdots, i_{k-1}})$ in $(a_{i_1, i_2, \cdots, i_{k-1}}, b_{i_1, i_2, \cdots, i_{k-1}})$ for every sequence $i_1, i_2, \cdots, i_{k-1}$ of positive integers.

(b) $\mathcal{C}_k = \bigcup_{i_1, i_2, \cdots, i_k = 1}^{\infty} \mathcal{C}_k^{i_1, i_2, \cdots, i_k}$, where $\mathcal{C}_k^{i_1, i_2, \cdots, i_k}$ is the collection of all subcontinua in I^{n+1} meeting both $\pi^{-1}(a_{i_1, i_2, \cdots, i_k})$ and $\pi^{-1}(b_{i_1, i_2, \cdots, i_k})$.

(c) $\alpha_k : \Delta_k \longrightarrow \mathcal{C}_k$ is a function such that

$$\alpha_k|_{\Delta(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k})} : \quad \Delta(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k}) \longrightarrow \mathcal{C}_k^{i_1, i_2, \cdots, i_k}$$

is a 1-1 correspondence for every sequence $i_1, i_2, \cdots, i_k$ of positive integers.

(d) For each $t \in \Delta_k$ let $y_t \in \pi^{-1}(t) \cap \alpha_k(t)$ and let $Y_k = \{y_t : t \in \Delta_k\}$.

By the construction we have the following property: For every nonempty interval $(a, b) \subset (0, 1)$, there exist integers $i_1, i_2, \cdots, i_k$ such that $(a_{i_1, i_2, \cdots, i_k}, b_{i_1, i_2, \cdots, i_k}) \subset (a, b)$.

Step 4. For every $t \in (0, 1) \setminus \bigcup_{k=0}^{\infty} \Delta_k$, we pick an arbitrary point $y_t \in \pi^{-1}(t)$ and put $Z_0 = \{y_t : t \in (0, 1) \setminus \bigcup_{k=0}^{\infty} \Delta_k\}$.

Finally, let $Z = Z_0 \cup \bigcup_{k=0}^{\infty} Y_k$.

Then $dim(Z) \geq n$. If $dim(Z) = n + 1$, by [H-W, Theorem IV.3, p.44], the set Z would contain a nonempty subset which is open in I^{n+1}. This is impossible since Z contains exactly one point from each hyperplane $\{y\} \times I^n$. Hence, $dim(Z) = n$. We shall show that Z is connected. If Z is not connected, then $Z = C \cup D$ where C and D are separated and nonempty. Let $c \in C$ and $d \in D$. By the Phragmen-Brouwer Theorem [Wi, Theorem 5.19, p.60] there exists a continuum E of $I^{n+1} \setminus (C \cup D)$ which separates c and d in I^{n+1}. Since I^{n+1} is an (n+1)-dimensional Cantor-manifold [H-W, Example VI.11, p.93], $\dim(E) \geq n$. Now, $\pi(E)$ is non-degenerate since otherwise E would contain $\pi^{-1}(t)$ for some $t \in (0,\ 1)$ which contradicts with $E \cap Z = \emptyset$. Let $(a,\ b) \subset \pi(E)$ for some $a < b$. Then, there exist integers $i_1, i_2, \cdots, i_k$ such that $(a_{i_1,i_2,\cdots,i_k},\ b_{i_1,i_2,\cdots,i_k}) \subset (a,\ b)$ and, hence, E meets both $\pi^{-1}(a_{i_1,i_2,\cdots,i_k})$ and $\pi^{-1}(b_{i_1,i_2,\cdots,i_k})$. This implies that E meets Y_k. This is a contradiction. So Z is connected. Since $|Z \cap \pi^{-1}(t)| = 1$ for each $t \in (0,\ 1)$, Z is a D_1-space. Therefore, the space Z is a connected, separable, metric D_1-space with $dim(Z) = n$. See Figure 1 below.

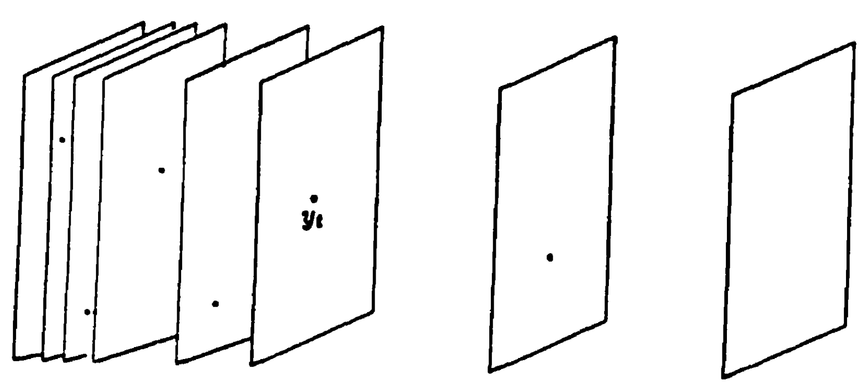

Figure 1 (for Y_0)

By gluing infinitely many of these sets into a chain we get an infinite dimensional example.

Remark. Note that for each integer m we can attach a simple m-od to Z to get a connected separable metric space with dimension n and disconnection number $m+1$. One can modify Gladdines' example X (Tymchatyn's description) by replacing each arc in X by a copy of the space Z in Example 6.1 to obtain a connected metric space with disconnection number $\aleph_0$ and and arbitrarily large finite dimension. By the results of Chapter 3 the space Z in Example 6.1 is homeomorphic to the real line in a coarser topology.

Example 6.2 *A locally connected separable metric space X with $D^s(X) = 1$ such that X is not rim-finite and $D^s(Y)$ is not defined for some connected subset Y of X.*

In the plane $\mathbf{R}^2$ denote $a_0 = (0,0)$ and $a_i = (1, \frac{1}{i})$ for $i > 0$. For each $i > 0$ denote $\overline{a_0a_i}$ the segment from a_0 to a_i. Let $X = \bigcup_{i=1}^{\infty}(\overline{a_0a_i} \setminus \{a_i\})$. Then X is a connected, hereditarily locally connected, separable metric space with $D^s(X) = 1$, but X is not rim-finite at the point a_0. Denote $b_i = (\frac{1}{2}, \frac{1}{2i})$ for $i > 0$. Then $Y = \bigcup_{i=1}^{\infty} \overline{a_0b_i}$ is a connected subset of X but $D^s(Y)$ is not defined. This example may be compared with Theorem 3.4.

Inspired by Example 6.2, we ask the following question.

Question 6.3 If X is a separable metric space with $D^s(X) \leq \aleph_0$ and Y is a subcontinuum of X, is there a countable subset C of Y such that $Y \setminus C$ is connected and $D^s(Y \setminus C) \leq \aleph_0$?

Remark. For locally connected separable metric spaces, the answer to Question 6.3 is positive because of the existence of a universal separable R-tree (see [MNO, Section 2]).

Question 6.4 *Let X be a Hausdorff hereditarily $D_{\aleph_0}$-space (see p.23).*

1. Is X the union of countably many subsets A_i's ($i \geq 0$) where A_0 is countable, A_i ($i > 0$) is connected and admits a one-to-one map into a generalized arc?

2. If A is closed and disconnects X, do components of $X \setminus A$ have interiors?

3. If A is closed and disconnects X, for all but finitely many components C of $X \setminus A$, does each point of C disconnect C?

4. If C is a component of $X \setminus \{p\}$, is p in the closure of C?

5. If A is closed in X and C is a component of $X \setminus A$, does there exist a connected subset C' of C such that C' is not separated from A and C' has no cutpoint?

6. Suppose A is a finite set of X not disconnecting X. Does there exist a finite set B containing A such that B is maximal with respect to not disconnecting X?

Question 6.5 *Let X be a metric continuum. What is the Borel class of the subspace $E_X(a,b)$ where $a,\ b \in X$?*

These subspaces may not be closed. By [Wh1, (5.1), p52] $E_X(a,b)$ is the union of a G_δ-set and a countable set. So $E_X(a,b)$ is $G_{\delta\sigma}$. Is it G_δ? It is known that $E_X(a,b)$ is closed if X is locally connected.

The following examples show inverse limits affect disconnection numbers and connectivity degree.

Example 6.6 *An inverse limit of D_4-spaces which is not a $D_{\aleph_0}$-space. This example is also an inverse limit of C_1-spaces which is a C_2-space. Our example is in fact an inverse limit of triods.*

We define $f:\ [0,\ 1] \longrightarrow [0,\ 1]$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}\ ; \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

For each positive integer i let X_i be the union of the graph of f^i and its reflection in the plane about the graph of $f^i|_{[0,\ \frac{1}{2^i}]}$. Thus X_i is a simple triod. Let $\mathcal{P} = \{X_i\}_{i=1}^{\infty}$

Then each $D^s(X_i) = 4$ and $C_m(X_i) = 1$. Let X be the union of $\{(x,\ \frac{3}{4} + \frac{1}{4}sin(\frac{1}{x})) \in \mathbf{R}^2 :\ 0 < |x| \le 1\}$ and the vertical segment from $(0,\ 1)$ to $(0,\ 0)$. X is not a $D_{\aleph_0}$-space since a countably infinite point set in the y-axis of X can not separate X. There exist two disjoint open sets joining $(0,\ 1)$ and $(0,\ \frac{1}{2})$. So $C_m(X) = 2$. For every $0 < \epsilon < 1$ it is easy to construct an ϵ-map of X onto X_i for i sufficiently large (See Figure 2 below). This implies X is $\mathcal{P}$-like and, hence, X is an inverse limit of a sequence in $\mathcal{P}$.

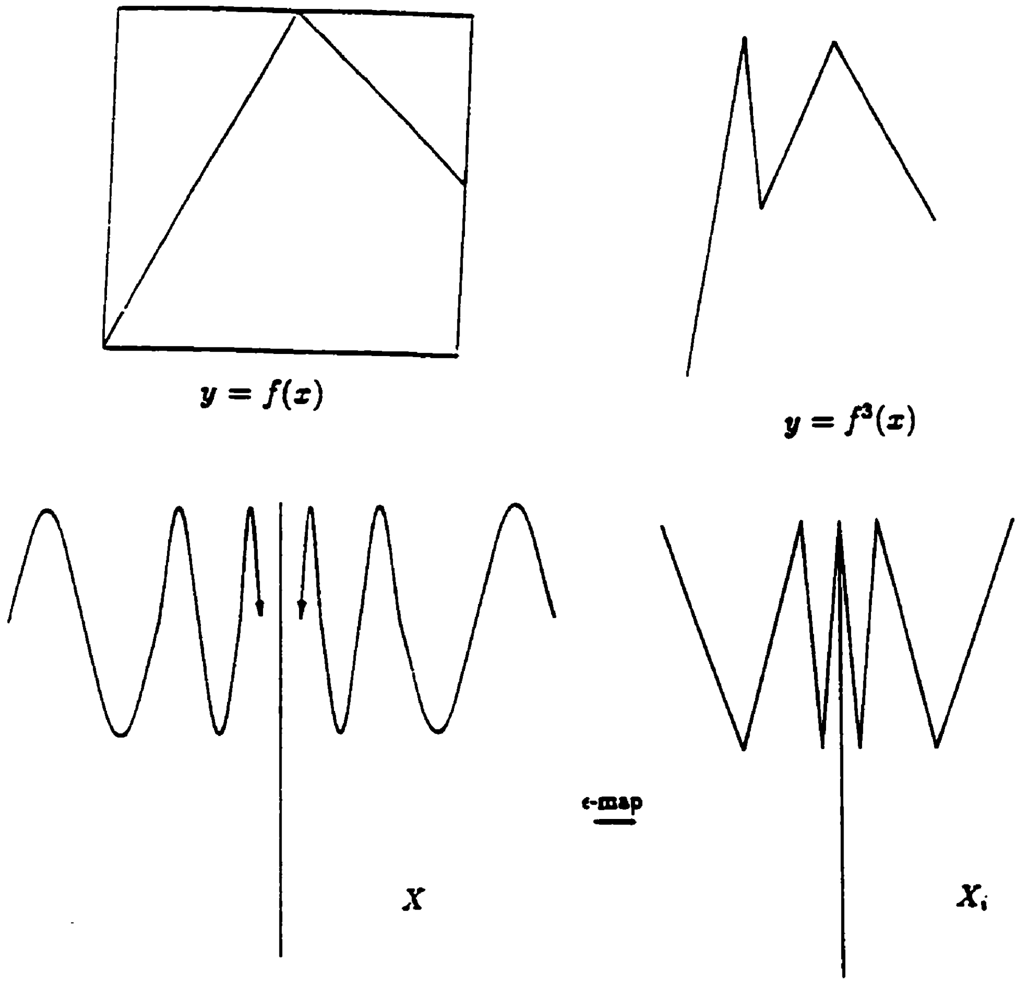

Figure 2

Example 6.7 *An inverse limit of D_4-spaces which is a D_3-space. This example is also an inverse limit of C_2-spaces which is a C_1-space.*

Let $\mathcal{P}_1$ be the set whose only element X is a simple triod and let $\mathcal{P}_2$ be the set whose only element Y is a simple closed curve with two stickers:

$$Y = \{(x,\ y) \in \mathbf{R}^2 : x^2 + y^2 = 1\} \cup \{(x,\ -1)\ :\ 0 \leq |x| \leq 1\}.$$

Then the element of $\mathcal{P}_1$ has disconnection number 4 and the element of $\mathcal{P}_2$ has connectivity degree 2. The unit interval $[0,\ 1]$ has disconnection number 3 and connectivity degree 1. It is both $\mathcal{P}_1$-like and $\mathcal{P}_2$-like: For every $0 < \epsilon < 1$ we identify the pair of points $\frac{1}{2} - x$ and $\frac{1}{2} + x$ for each $0 \leq x \leq \frac{\epsilon}{4}$ in $[0,\ 1]$. Then the quotient space of $[0,\ 1]$ is homeomorphic to $X \in \mathcal{P}_1$ and the quotient map is an ϵ-map. Hence, $[0,\ 1]$ is $\mathcal{P}_1$-like. Similarly, for every $0 < \epsilon < 1$ we only identify the pair of points $\frac{1}{2} - \frac{\epsilon}{4}$ and $\frac{1}{2} + \frac{\epsilon}{4}$ in $[0,\ 1]$. Then the quotient space of $[0,\ 1]$ is homeomorphic to $Y \in \mathcal{P}_2$ and the quotient map is an ϵ-map. Hence, $[0,\ 1]$ is $\mathcal{P}_2$-like. By the $\mathcal{P}$-like Theorem (1.5.6), $[0,\ 1]$ is an inverse limit both in $\mathcal{P}_1$ and $\mathcal{P}_2$

Example 6.8 *An inverse limit of $D_{\aleph_0}$-spaces which is not a $D_{\aleph_0}$-space. This example is also an inverse limit of finite connectivity degree spaces which is not a finite connectivity degree space.*

In the plane $\mathbf{R}^2$ we define $L_0 = [-1,\ 1] \times \{0\}$ and for each $i \geq 1$ we define

$$L_i = \{(x,\ y) \in \mathbf{R}^2 :\ x^2 + (y - i)^2 = i^2 + 1 \text{ and } y \leq 0\}.$$

For each $i \geq 0$ let $X_i = \bigcup_{j=0}^{i} L_j$ and let f_i be a natural retraction of X_{i+1} to X_i by pushing L_{i+1} onto L_0. Then (X_i, f_i) is an inverse sequence with each $D^s(X_i) < \aleph_0$ and $C_m(X_i) < \aleph_0$. By the Anderson-Choquet Embedding Theorem (1.5.5),

$$\varprojlim\ (X_i, f_i) = \bigcup_{i=0}^{\infty} L_i$$

without disconnection number and its connectivity degree is not finite.

Example 6.9 *An inverse limit of $D_{\aleph_0}$-spaces which is not a $D_{\aleph_0}$-space even though the bonding mappings are monotone.*

In the plane $\mathbf{R}^2$ let $O = (0,\ 0)$. For each $i \geq 0$ let

$$S_i = \{(x,\ y) \in \mathbf{R}^2 :\ (x - \tfrac{1}{i})^2 + y^2 = \tfrac{1}{i}^2\}.$$

Let $X_i = \bigcup_{j=1}^{i} S_j$ and let f_i be the monotone retraction of X_{i+1} to X_i which shrinks the circle S_{i+1} into the point O. Then (X_i, f_i) is an inverse sequence with each $D^s(X_i) < \aleph_0$ and the bonding mappings are monotone. Again by the Anderson-Choquet Embedding Theorem $\varprojlim\ (X_i, f_i) = \bigcup_{i=1}^{\infty} S_i$, the Hawaiian Earring, whose disconnection number is not defined.

Example 6.10 *There exists a metric space X with $D^s(X) = \aleph_0$ but there exist an uncountable number of independent connections between some two points of X. This shows that the local connectivity assumption in Theorem 5.4 is necessary.*

We modify Pierce's example [Pi]. Let W be the set of all countable (including finite) ordinal numbers with the discrete topology, and let $A = \{(x,\ sin(\frac{1}{x})) \in \mathbf{R}^2\text{: } 0 < x \leq 1\}$ the *open $sin(\frac{1}{x})$-curve*. Let $\Phi = \{F_\alpha\}_{\alpha < \aleph_1}$ be a partition of W such that for each $\alpha < \aleph_1$

$|F_\alpha| = n$ for some positive integer n. Let $\Pi = \{P_\beta\}_{\beta<\aleph_1}$ be the family of all those two point subsets of W which intersect two members of Φ. For each $\beta < \aleph_1$ let A_β be a copy of A with the two points of P_β as its only limits and such that $A_\beta \cap A_\gamma = \emptyset$ for $\beta \neq \gamma$. Define X to be $W \cup \bigcup_{\beta<\aleph_1} A_\beta$. Then every infinite subset of X separates X. There exist an uncountable number of independent connections between the two points of P_β for each $\beta < \aleph_1$. A metric is easily introduced as in Gladdines' example (Tymchatyn's description).

Example 6.11 *Let X be the space obtained by adding end points of all the segments in Example 6.2 then X is a locally connected, separable metric space with $D^s(X) \not\leq \aleph_0$ but $C_m(X) = 1$.*

Example 6.12 *There exists a separable metric space X such that X has finite connectivity degree but there exist two points of X which can not be separated by any finite subset of X. Thus the local connectivity assumption in Theorem 5.3 is necessary.*

Let X be the Warsaw circle in $\mathbf{R}^2$ which is the union of the closure of the set $\{(x,\ sin(\frac{1}{x}) \in \mathbf{R}^2 :\ 0 < x \leq 1\}$ and three convex arcs, one from $(0,\ -1)$ to $(0,\ -2)$, one from $(0,\ -2)$ to $(1,\ -2)$, one from $(1,\ -2)$ to $(1,\ sin(1))$. Then $C_m(X) = 2$. The two points $(0,\ 0)$ and $(0,\ 1)$ can not be separated by any finite subset of X. It follows that Theorem 5.3 fails for non-locally connected continua. We note that X is 2-point connected between $(0,\ 1)$ and $(1,\ sin(1))$ but there do not exist two independent connections between them. This gives a negative answer to a question in [Tym] which said 'if X is a regular, T_1 space and P and Q are disjoint closed sets in X such that X is n-point strongly connected between P and Q, do there exist disjoint open sets $U_1, \cdots, U_n$ such that U_i cannot be separated between P and Q?' In other words, the n-open connections theorem fails for non-locally connected spaces.

The following is a higher dimension disconnection problem.

Question 6.13 *Suppose X is a connected, locally connected, complete, metric space which is disconnected by the removal of any $\aleph_0$ disjoint simple closed curves. What can one say about the space X?*

If one requires that each simple closed curve disconnect one has characterizations of the 2-sphere and of 2-manifolds, respectively, as follows.

Bing's Theorem ([Bing], p.646) *If no pair of points of a locally connected metric continuum S separates it, but every simple closed curve in S does separate it, then S is a 2-sphere.*

van Kampen's Theorem ([Yo], Theorem 1.1, p.979) *Let X be a non-degenerate, locally compact, locally connected, connected, metric space with no local separating points. Suppose that for each point x of X there is a neighborhood U of x such that every simple closed curve in U separates X. Then X is a 2-manifold.*

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